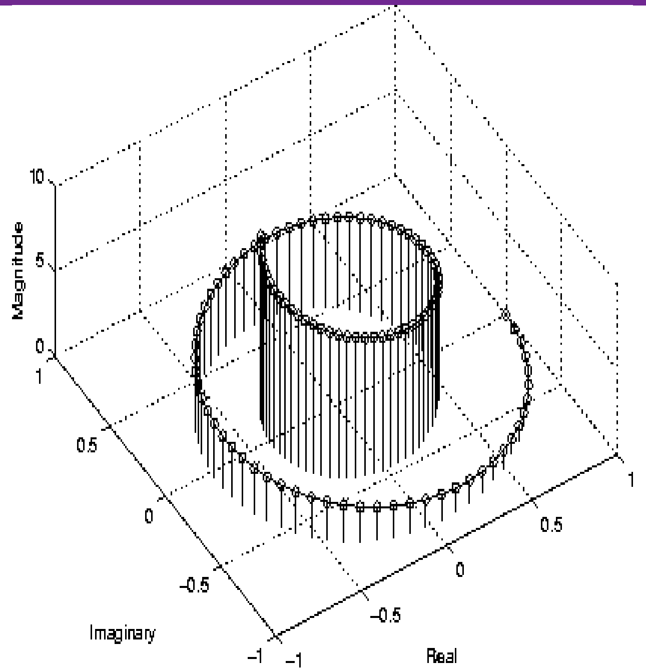
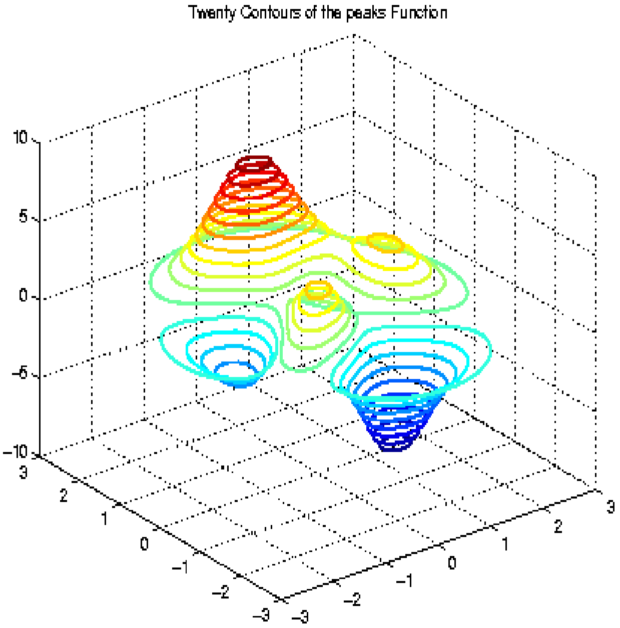
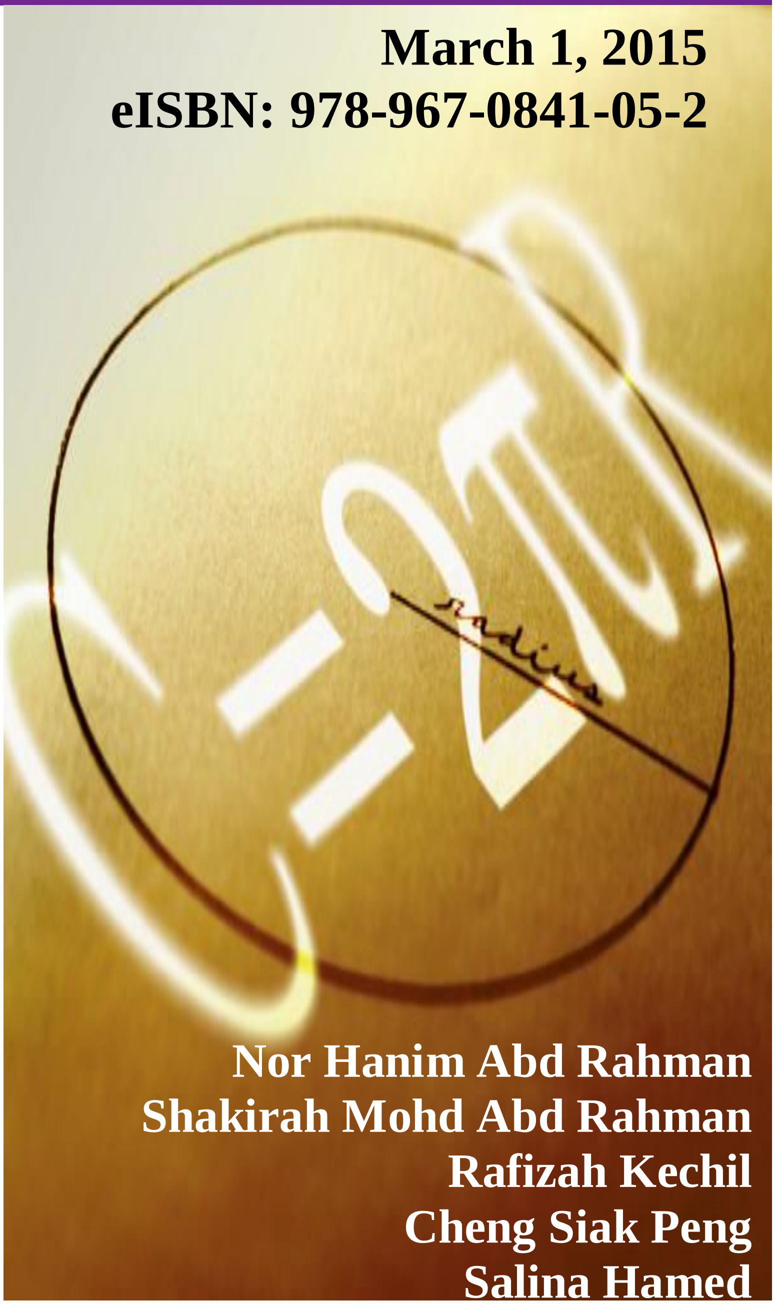
FURTHER MATHEMATICS FOR SCIENCE AND ENGINEERING







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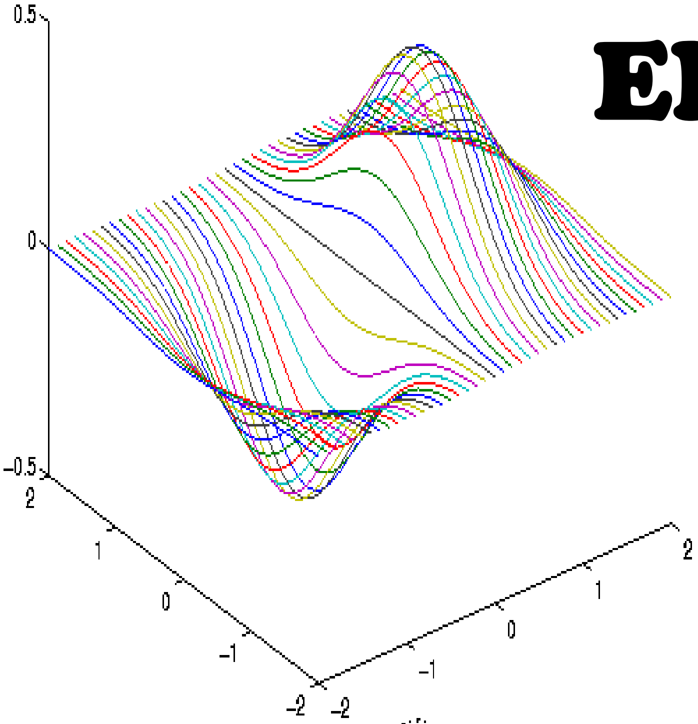

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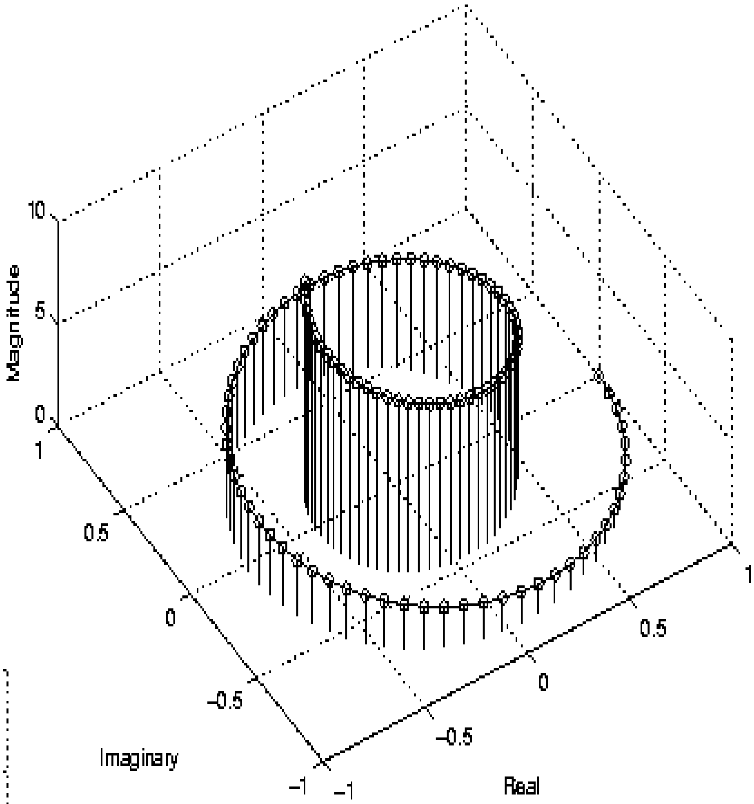

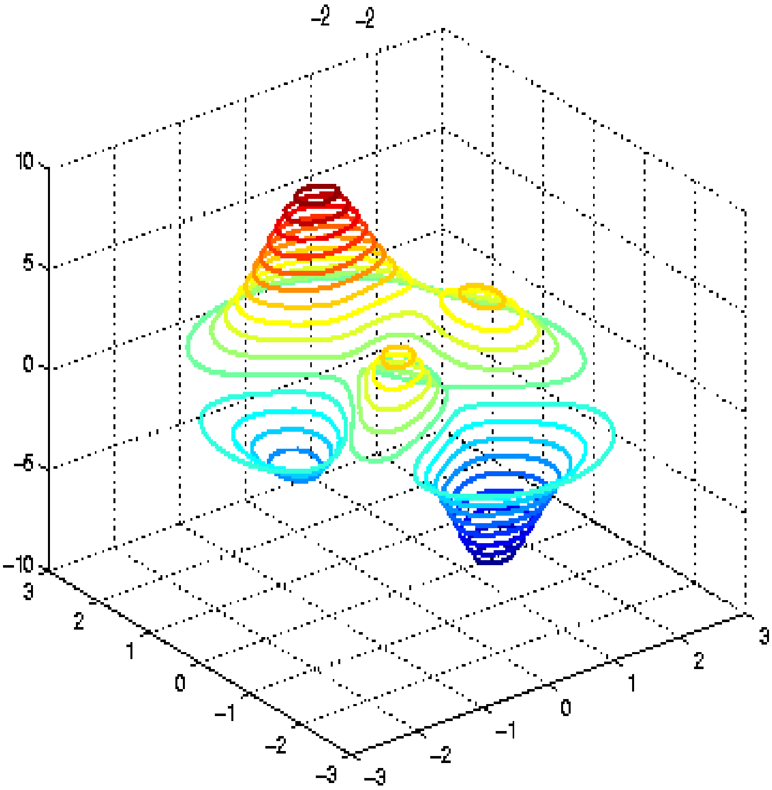

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D E D I C A T I O N

To Our Families, Friends, Colleagues & Students

To Our Parents --

Thanks for your encouragement, love and inspiration.

CONTENT

TUTORIAL

Chapter One

MATRIX ALGEBRA

CHAPTER 1: MATRIX ALGEBRA

Definition:
A set of real or complex numbers arranged in a rectangular array having *m* rows and *n* columns. This set is called *a matrix of order m x n* (**read as m by n**).

The number $\mathbf{a_{ij}}$ is called an ***element*** in row i and column j.

The capital letters are used to denote matrices and the corresponding lower case letters are used to denote the elements.

The matrix A can also be abbreviated by (a_{ij}) or $[a_{ij}]$

Notation:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ or } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Definition:
Equality of Matrices
Two matrices A = (a_{ij}) and B = (b_{ij}) are said to be equal if they have the same order and their corresponding elements are equal.

Example 1:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 5 & 0 \end{pmatrix}$. Find the values for a, b, c and d.

Solution:

a = 1, b = -4, c = 5, d = 0

Example 2:

Find the values of a, b, c and d such that the matrices $A = \begin{pmatrix} a+3 & 2b+a \\ c-1 & 4d-6 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -7 \\ 3 & 2d \end{pmatrix}$ are equal.

Solution:

$a + 3 = 0 \quad \Rightarrow \quad a = -3$

$2b + a = -7 \quad \Rightarrow \quad b = \dfrac{-7-(-3)}{2} = -2$

$c - 1 = 3 \quad \Rightarrow \quad c = 4$

$4d - 6 = 2d \quad \Rightarrow \quad d = 3$

SPECIAL MATRICES:

1) Square Matrix

A square matrix is a matrix having same number of rows and columns.

Example:

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}_{2\times 2}, \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 5 & 4 \\ 2 & 5 & 1 \end{bmatrix}_{3\times 3}$$

2) Column and Row matrix

The matrix of order 1 x n is called ***a row matrix***, while a matrix of order m x 1 is called ***a column matrix***.

Example:

i) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}, \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}_{4\times 1}$ **column matrix**

ii) $\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}_{1x3}, \begin{bmatrix} 2 & 1 & 3 & 7 \end{bmatrix}_{1x4}$ **row matrix**

3) Null or Zero Matrix

A matrix with all its elements equal to zero. It is denoted by the letter **'O'**.

Example:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = O, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = O$$

4) Identity Matrix

A square matrix A = (aij) of the order n x n with $a_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$. is called an identity matrix of order n and is denoted by I_n.

Example:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

The identity matrices have the property of 1 in elementary algebra, i.e. AI = A and IB=B, whenever the products are defined.

5) Diagonal Matrix

A square matrix with all the elements below and above the main diagonal are zeroes.

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

6) Triangular Matrix

A square matrix with the elements below or above the main diagonal are zeroes.

Example:

i) $\begin{bmatrix} 4 & 1 & 2 \\ 0 & 5 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ **Upper** triangular matrix

ii) $\begin{bmatrix} 4 & 0 & 0 \\ 1 & 5 & 0 \\ 2 & 5 & 1 \end{bmatrix}$ **Lower** triangular matrix

7) Transposed Matrix

A^T, can be obtained by interchanging rows and columns of a matrix A.

Example:

$$A = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 4 & -7 \end{bmatrix}_{3\times 2} \xrightarrow{\text{transpose}} A^T = \begin{bmatrix} 3 & 2 & 4 \\ 1 & -1 & -7 \end{bmatrix}_{2\times 3}$$

8) Symmetric Matrix

A^T=A, then A is said to be a symmetric matrix.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \xrightarrow{\text{transpose}} A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

9) Augmented Matrix

A matrix formed by placing two matrices side by side and a vertical line is used to distinguish the matrices.

Example:

i) $\left[\begin{array}{ccc|c} 1 & 5 & 2 & 1 \\ -2 & 3 & 1 & 2 \\ 0 & 5 & 7 & 5 \end{array}\right]$,

ii) $\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 2 & 3 \\ -2 & 3 & 1 & 2 & 4 & 5 \\ 0 & 5 & 7 & 3 & 5 & 6 \end{array}\right]$

OPERATIONS ON MATRICES:

A. Scalar Multiplication

If A = (a_{ij}) is a matrix of order m x n and ka is a scalar, the product of A by k is kA = (ka_{ij}).

Properties of a scalar multiplication:

1. 0A =A0 = O
2. p(qA) = (pq) A
3. (-1) A = -A

Example 1:

If $A = \begin{pmatrix} 3 & -5 & 1 \\ 0 & 2 & 7 \end{pmatrix}$, find 2A, $-3A$ and $\frac{1}{2}A$.

Solution:

i) $2A = \begin{pmatrix} 6 & -10 & 2 \\ 0 & 4 & 14 \end{pmatrix}$

ii) $-3A = \begin{pmatrix} -9 & 15 & -3 \\ 0 & -6 & -21 \end{pmatrix}$

iii) $\frac{1}{2}A = \begin{pmatrix} \frac{3}{2} & \frac{-5}{2} & \frac{1}{2} \\ 0 & 1 & \frac{7}{2} \end{pmatrix}$

B. Addition and Subtraction

If A = (a_{ij}) and B = (b_{ij}) are two m x n matrices, then

(i) A + B = ($a_{ij} + b_{ij}$)
(ii) A – B = ($a_{ij} - b_{ij}$)

Note: The addition and subtraction of two matrices can only be done when **both matrices are of the same order**.

Basic properties of addition and scalar multiplication of matrices:

1. A + B = B + A - commutative law
2. (A + B) + C = A + (B + C) - associative law
3. A + O = O + A = A
4. k(A + B) = kA + kB - distributive law
5. IA = AI = A
6. (p + q)A= pA + qA, where k,p,q are scalars - distributive law
7. A + (-A) = O

Example 1:

If $A = \begin{pmatrix} 1 & -2 & 4 \\ -3 & 1 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 4 & -2 \\ 1 & 0 & 5 \end{pmatrix}$ *and* $C = \begin{pmatrix} -6 & 1 & 3 \\ 5 & 0 & 2 \end{pmatrix}$, find

a) A + B
b) A – C
c) 7B- 4A
d) 2A – 5B + $\frac{1}{3}C$

Solution:

a) A + B = $\begin{pmatrix} 1 & -2 & 4 \\ -3 & 1 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 4 & -2 \\ 1 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ -2 & 1 & 10 \end{pmatrix}$

b) A – C= $\begin{pmatrix} 1 & -2 & 4 \\ -3 & 1 & 5 \end{pmatrix} - \begin{pmatrix} & & \\ & & \end{pmatrix} = \begin{pmatrix} & & \\ & & \end{pmatrix}$

c) 7B- 4A = $7\begin{pmatrix} & & \\ & & \end{pmatrix} - 4\begin{pmatrix} & & \\ & & \end{pmatrix} = \begin{pmatrix} & & \\ & & \end{pmatrix}$

d) 2A – 5B + $\frac{1}{3}C$ =

Example 2:

If $3A + B = 2C$, where $A = \begin{pmatrix} a & -a \\ c & b \end{pmatrix}$, $B = \begin{pmatrix} -b & c \\ a & -a \end{pmatrix}$, and $C = \begin{pmatrix} 4 & -3 \\ 3 & -4 \end{pmatrix}$. Determine the values of a, b and c.

Solution:

C. Matrix multiplication

If $A=(a_{ij})$ is an mxn matrix while $B=(b_{ij})$ is an nxp matrix, the product AB is a matrix $C=(c_{ij})$ of order mxp where

$$C_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj} \quad \text{for all } i = 1,2,...,m \quad \text{and} \quad j = 1,2,.....,p$$

Note: The matrix multiplication AB is defined if and only if the **number of columns of A is equal to the number of rows of B**. Such matrices are called ***comformable*** for matrix multiplication.

Properties of matrix multiplication:

1. $k(AB) = (kA)B = A(kB) = ABk$, where k = scalar
2. $A(BC) = (AB)C$
3. $(A+B)C = CA + CB$
4. $C(A+B) = CA + CB$
5. $AB \neq BA$
6. If $A = O$ or $B = O$, then $AB = O$
   But conversely, $AB = O$ does not imply $A = O$ or $B = O$.
7. If $AB = AC$, it does not imply that $B = C$
   (Cancellation law is not valid in matrix multiplication)

Example 1:

Given the matrices:

$$A = \begin{pmatrix} 2 & -3 & 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 2 \\ -1 \\ -2 \\ 5 \end{pmatrix},\ C = \begin{pmatrix} 2 & -1 & 3 & 1 \\ 4 & 2 & 1 & 0 \end{pmatrix},$$

Find the products of AB, BA, BC and CB.

Solution:

Example 2:

If $A=\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$, $B=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$, $C=\begin{pmatrix}2 & 1 & -1\\ 0 & 2 & 6\end{pmatrix}$ and $D=\begin{pmatrix}2 & -2 & 1 & 0\\ 4 & 1 & 0 & 2\\ 0 & 3 & 1 & -1\end{pmatrix}$

Verify (i) A(B+C) = AB + AC

(ii) (AB)D = A(BD)

Solution:

Example 3:

If A, B and C are three matrices such that AC=CA and BC=CB, show that (AB+BA)C=C(AB+BA).

Solution:

Example 4:
Simplify the following expression:
(a) A (3B- C) + (A-2B)C + 2B (C + 3A)
(b) A (BC-CD) + A(C-B) D – AB(C-2D)

Solution:
(a) A (3B- C) + (A-2B)C + 2B (C + 3A)

$= 3AB \cancel{-AC} \cancel{+AC} \cancel{-2BC} + 2BC + 6BA$

$= 3AB - 6BA$

(b) A (BC-CD) + A(C-B) D – AB(C-2D)

$= \cancel{ABC} \cancel{-ACD} + \cancel{ACD} - ABD \cancel{-ABC} + 2ABD$

$= ABD$

D. Powers of a matrix

The integral powers $A^0,\ A^1, A^2, A^3, \ldots,\ A^n$ of a square matrix A are defined as follows

$A^0 = I,$

$A^1 = A,$

$A^2 = A.A,$

$A^3 = A.A.A = A.A^2,$

$\vdots$

$A^n = A.A.A......A_n$

The law of exponents is valid for powers of matrix. i.e.

$A^pA^q = A^{p+q}, \quad (A^p)^q = A^{pq} \qquad \text{for} \quad p > o, \text{ and } q > 0$

Example 1:
Show that $2I - 11A + 7A^2 - A^3 = O$.

Solution:

$$A^2 = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 2 & 0 \\ 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 3 \\ -1 & 2 & 0 \\ 1 & 0 & 4 \end{pmatrix}$$

E. Transpose of a matrix

Let A be an m x n (non-square) matrix. Then the **transpose** of a matrix A is written by $\mathbf{A^T}$ **or A'** is the n x m matrix obtained by interchanging the rows and columns of A.

(Recall: $\mathbf{A^T}$ = **m x n** matrix ➔ **n x m** matrix)

***Properties of transposition*:**

1. $(A^T)^T = A$
2. $(kA)^T = kA^T$, where k is a scalar
3. $(A + B)^T = A^T + B^T$
4. $(AB)^T = B^T A^T$
5. $(AB)^T \neq A^T B^T$

Example 1:

If $A = \begin{pmatrix} 1 & 3 & -2 & 5 \\ 1 & 7 & 9 & -2 \\ 0 & 11 & 0 & -5 \end{pmatrix}$, find A^T

Solution:

Just need to Interchange 'row' to 'column', so

$$A^T = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 7 & 11 \\ -2 & 9 & 0 \\ 5 & -2 & -5 \end{pmatrix}$$

Example 2:

If $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ and $B = \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix}$, show that $(A + B)^T = A^T + B^T$.

Solution:

Example 3:

Given A = $\begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$ and B = $\begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix}$. Show that the matrix $(AB)^T = B^T A^T \neq A^T B^T$

Solution:

$$(AB) = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} a11 & a12 \\ a21 & a22 \end{pmatrix}$$

$$= \begin{pmatrix} 6(2)+(-3)(1) & 6(4)+(-3)(-1) \\ (-2)(2)+(1)(1) & (-2)(4)+(1)(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 27 \\ -3 & -9 \end{pmatrix}$$

$\Rightarrow$ $\mathbf{(AB)^T} = \begin{pmatrix} 9 & -3 \\ 27 & -9 \end{pmatrix}$

$\Rightarrow$ $\mathbf{B^T A^T} = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 6 & -2 \\ -3 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 2(6)+(1)(-3) & 2(-2)+1(1) \\ (4)(6)+(-1)(-3) & (4)(-2)+(-1)(1) \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -3 \\ 27 & -9 \end{pmatrix}$$

$\Rightarrow$ $\mathbf{A^T B^T} = \begin{pmatrix} 6 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 6(2)+(-2)(4) & 6(1)+(-2)(-1) \\ (-3)(2)+(1)(4) & (-3)(1)+(1)(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 9 \\ -2 & -4 \end{pmatrix}$$

Therefore, $(AB)^T = B^T A^T \neq A^T B^T$.

DETERMINANT

Determinants are very useful in the analysis and solution of systems of linear equations. A non-homogeneous system of linear equations has a unique solution if and only if the determinant of the system's matrix is nonzero (i.e., the matrix is **nonsingular**). Determinants are defined only for square matrices.

If the determinant of a matrix is 0, the matrix is said to be singular otherwise is non-singular matrix.

Notation :

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \text{or} \quad \det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

A. 2x2 Matrix

Determinant for 2 x 2 matrix is defined to be

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

B. 3x3 Matrix

Sarrus's rule (only for 3×3 matrix)

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$|A| = \begin{vmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{vmatrix}$$

= (aei+bfg+cdh)-(ceg+afh+bdi)

Example 1:

Calculate the determinant of matrix A= $\begin{bmatrix} 8 & 3 & 1 \\ 7 & -2 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ by using Sarrus's Rule:

Solution:

$$|A| = \left[\begin{matrix} 8 & 3 & 1 \\ 7 & -2 & 2 \\ 0 & 1 & 3 \end{matrix}\right]\begin{matrix} 8 & 3 \\ 7 & -2 \\ 0 & 1 \end{matrix}$$

$$= -(0+16+63) + (-48+0+7)$$

$$= -79 - 41 = -120$$

C. nxn Matrix

Laplace expansion

A nxn determinant can be expanded "by minors" to obtain

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix} + \ldots \pm a_{1n}\begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(n-1)} \end{vmatrix}$$

A general determinant for a matrix A has a value

$$|A| = \sum_{i=1}^{n} a_{ij}C_{ij}$$

Where C_{ij} is the cofactor of a_{ij} defined by

$$C_{ij} = (-1)^{i+j}M_{ij}$$

and M_{ij} is the **minor** of matrix A formed by eliminating row i and column j from matrix A.

Example 1: (Finding Minor)

Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, find the minors for a_{11}, a_{12} and a_{32}

Solution:

$M_{11} = \begin{vmatrix} e & f \\ h & i \end{vmatrix} = ei - hf$ eliminate row 1 and column 1

$M_{12} = \begin{vmatrix} d & f \\ g & i \end{vmatrix} = di - gf$ eliminate row 1 and column 2

$M_{32} = \begin{vmatrix} a & c \\ d & f \end{vmatrix} = af - cd$ eliminate row 3 and column 2

Example 2: (Finding Minor and Cofactor)

Given : $A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 6 \\ -1 & 2 & 0 \end{pmatrix}$

Find minors for M_{11} , M_{23} , M_{33} , M_{31} and cofactors for C_{11}, C_{23} , C_{33} , C_{31} .

Solution:

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 6 \\ -1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$M_{11} = \begin{vmatrix} 3 & 6 \\ 2 & 0 \end{vmatrix} = 0 - 12 = -12$ ➔ $C_{11} = (-1)^{1+1}(-12) = +(-12) = 12$

$M_{23} = \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} = 2 - (-2) = 4$ ➔ $C_{23} = (-1)^{2+3}(4) = -(4) = -4$

$M_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1$ ➔ $C_{33} = (-1)^{3+3}(-1) = -(-1) = 1$

$M_{31} = \begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} = 12 - 15 = -3$ ➔ $C_{31} = (-1)^{3+1}(-3) = +(-3) = -3$

Laplace Expansion

Finding determinant by using Laplace Expansion for nxn matrix:

$$\textbf{Minor of A} = M_{ij} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & .. & a_{1j\,j} & .. & a_{1n} \\ a_{21} & a_{22} & a_{23} & .. & a_{2j} & .. & a_{2n} \\ a_{31} & a_{32} & a_{33} & .. & a_{3j} & .. & a_{3n} \\ .\ : & : & : & & : & & : \ : \\ a_{i1} & a_{i2} & a_{13} & .. & a_{ij} & .. & a_{1n} \\ : & : & : & & : & & : \\ a_{n1} & a_{n2} & .. & .. & a_{nj} & .. & a_{nn} \end{vmatrix}$$

$= a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + a_{i4}C_{i4} + \ldots + a_{in}C_{in}$ **(along ith row)**

or

$= a_{1j}C_{1j} + a_2jC_{2j} + a_{3j}C_{3j} + a_4jC_{4j} + \ldots + a_njC_{nj}$ **(along jth column)**

Example 1:

Calculate the determinant of matrix A=$\begin{bmatrix} 8 & 3 & 1 \\ 7 & -2 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ by using Laplace expansion:

(i) using row one
(ii) using column one
(iii) using row three

Solution:

(i) $|A| = 8\begin{vmatrix} -2 & 2 \\ 1 & 3 \end{vmatrix} - 3\begin{vmatrix} 7 & 2 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ 0 & 1 \end{vmatrix}$

$= 8(-6-2) - 3(21-0) + (7-0)$

$= -64 - 63 + 7$

$= -120$

(iii) $|A| = -\begin{vmatrix} 8 & 1 \\ 7 & 2 \end{vmatrix} + 3\begin{vmatrix} 8 & 3 \\ 7 & -2 \end{vmatrix}$

$= -(16-7) + 3(-16-21)$

$= -9 - 111$

$= -120$

(ii) $|A| = 8\begin{vmatrix} -2 & 2 \\ 1 & 3 \end{vmatrix} - 7\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$

$= 8(-6-2) - 7(9-1)$

$= -64 - 56$

$= -120$

Example 2:

By using (i) Laplace Expansion (ii) Sarrus's Rule , calculate the determinant of A where

$$A = \begin{pmatrix} 7 & 1 & 0 \\ 3 & -5 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution:

Properties of the Determinant

1. If k is a scalar and A is an nxn square matrix, then

$$|kA| = k^n |A|$$

Example 1:

Let matrix A = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 1 & -2 \end{pmatrix}$.

Calculate : (i) det A
(ii) 5 det (A)
(iii) det (5A)
(iv) 5^3 det(A)

Show that |5A| is equal to 5^n |A|, where n is the number of column.

Solution:

2. Determinants are **distributive, so**

$$|AB| = |A||B|$$

Example 2:

Given matrix $A = \begin{bmatrix} 1 & 5 \\ -7 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$. Find

(i) $|A|$
(ii) $|B|$
(iii) Product of matrix A and B
(iv) $|AB|$

Hence, show that $|AB| = |A||B|$

Solution:

3. If matrix inverse of A is A^{-1}**,** then

$$\left|A^{-1}\right| = \frac{1}{|A|}$$

Example 3:

Given A = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 1 & -2 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} -\frac{4}{3} & \frac{7}{12} & -\frac{1}{4} \\ \frac{2}{3} & -\frac{1}{6} & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{12} & -\frac{1}{4} \end{pmatrix}$. Calculate |A| and $\left|A^{-1}\right|$.

Hence show that $|A| = \dfrac{1}{|A^{-1}|}$

Solution:

4. If B is obtained from A by interchanging two different rows (columns), then the sign of the determinant is changed.

$$|B| = -|A|$$

Example 4:

If $A = \begin{pmatrix} 7 & 9 \\ -3 & 1 \end{pmatrix}$,

(i) Find the |A|
(ii) Let matrix B be a matrix resulting from interchanging column1 with column 2 of matrix A, then find the |B|
(iii) Let matrix C be a matrix resulting from interchanging row1 with row 2 of matrix B, then find the |C|
(iv) Show that |A| = -|B| = |C|

Solution:

5. If any two rows (or columns) of a matrix A are identical or multiple of each other, then

$$|A| = 0$$

Example 5:

(i) Let matrix A = $\begin{pmatrix} \boxed{1 \quad 2 \quad 3} \\ 4 \quad 5 \quad 6 \\ \boxed{1 \quad 2 \quad 3} \end{pmatrix}$, then $|A| = 0$

(ii) Let matrix G = $\begin{pmatrix} 1 & 2 & 1 \\ 4 & 5 & 4 \\ 1 & 2 & 1 \end{pmatrix}$, then $|G| = 0$

(iii) Let matrix H = $\begin{pmatrix} \boxed{1 \quad 2 \quad 1} \\ \boxed{2 \quad 4 \quad 2} \\ 0 \quad 1 \quad 5 \end{pmatrix}$, then $|H| = 0$

6. If B is obtained from A by multiplying a row (column) by scalar k , then

$$|B| = k|A|$$

Example 6:

Simplify the following determinant:

(i) $\begin{vmatrix} a & b & c \\ kd & ke & kf \\ g & h & i \end{vmatrix}$

(ii) $\begin{vmatrix} 2 & 4 & 6 \\ kd & ke & kf \\ -3 & 6 & -9 \end{vmatrix}$

Solution:

(i) $\begin{vmatrix} a & b & c \\ kd & ke & kf \\ g & h & i \end{vmatrix} = k \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ k factored from row 2

(ii) $\begin{vmatrix} 2 & 4 & 6 \\ kd & ke & kf \\ -3 & 6 & -9 \end{vmatrix} = (2)(k)(3) \begin{vmatrix} 1 & 2 & 3 \\ d & e & f \\ -1 & 2 & -3 \end{vmatrix}$ 2 factored from row 1
k factored from row 2
3 factored from row 3

7. If B is obtained from A by operation ‘adding a multiple of a row (column) to a different row (column’, then

$$|B| = |A|$$

where the operation can be illustrated as

$R_j = kR_i + R_j$

$C_j = kC_i + C_j$

Example 7:

Given a matrix $A = \begin{bmatrix} 2 & 1 \\ -9 & 4 \end{bmatrix}$**.** Find

(i) $|A|$

(ii) Matrix B obtained by doing operation $R_2 = 2R_1 + R_2$

(iii) $|B|$

Hence, show that $|B| = |A|$.

8. if every element of a row or a column of determinant |A| is the sum of two terms, then |A| can be expressed as the sum of two determinants

Example 8:

(i) $$\begin{vmatrix} a+2 & 4+b & d-5 \\ 0 & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & d \\ 0 & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 2 & 4 & -5 \\ 0 & e & f \\ g & h & i \end{vmatrix}$$

(ii) $$\begin{vmatrix} -4 & 8 & -3 \\ 0 & e & f \\ 5 & -2 & 3 \end{vmatrix} + \begin{vmatrix} -4 & 8 & -3 \\ 0 & e & f \\ g & 6 & 1 \end{vmatrix} = \begin{vmatrix} -4 & 8 & -3 \\ 0 & e & f \\ 5+g & 4 & 4 \end{vmatrix}$$

9. The determinant of a matrix transpose equals the determinant of the original matrix,

$$|A| = |A^T|$$

Example 9

Calculate the |B| and $|B^T|$ for the given matrix B = $\begin{vmatrix} 3 & 1 & 6 \\ 1 & 0 & 2 \\ 4 & -1 & -2 \end{vmatrix}$

Solution:
(hint: show that $|B|=|B^T|$)

10. If A is a **triangular matrix**, then the |A| is the product of the elements on the main diagonal, such that

$$|A| = (a_{11})(a_{22})(a_{33})\ldots.(a_{nn})$$

Example 10:

Evaluate the following determinants using the properties.

a) $\begin{vmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i+kh \end{vmatrix}$

Solution:

MORE EXAMPLES ON FINDING DETERMINANTS USING LAPLACE EXPANSION, PROPERTIES AND "SARRUS'S RULE"

Example 1:

By using (i) Laplace Expansion
(ii) Sarrus's Rule (arrows)
(iii) Properties of determinant

calculate the determinant of A where

$$A=\begin{pmatrix} 2 & 1 & 0 \\ 4 & 5 & -1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Solution:

(i) Choose any row or column with the most numbers of zero i.e. Row 1 or column 3; then, identify the cofactor of each element➔ $(-1)^{i+j}\,M_{ij}$:

Say expanding along row 1 :

$$|A| = (+)\,2\begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} + (-)\,1\begin{vmatrix} 4 & -1 \\ 1 & 2 \end{vmatrix}$$

$= 2(10-(-1)) \quad + \quad (-1)(8-(-1))$
$= 2(11) + (-1)(9)$
$= 22 - 9$
$= 13$

(ii) This only applies for 3x3 matrices. Rewrite the first two columns in front of the third column such that :

$$\begin{array}{ccc|cc} 2 & 1 & 0 & 2 & 1 \\ 4 & 5 & -1 & 4 & 5 \\ 1 & 1 & 2 & 1 & 1 \end{array}$$

(-)

(+)

$|A| = [\ 2(5)(2)+(1)(-1)(1)+0(4)(1)\ ] - [\ 1(5)(0)+(1)(-1)(2)+2(4)(1)\ ]$
$= 20 - 1 + 0 - 0 + 2 - 8$
$= 13$

(iii) Convert the given matrix in form of triangular matrix

$$\begin{vmatrix} 2 & 1 & 0 \\ 4 & 5 & -1 \\ 1 & 1 & 2 \end{vmatrix} \xrightarrow{\text{Transpose}}$$

$$= \begin{vmatrix} 2 & 4 & 1 \\ 1 & 5 & 1 \\ 0 & -1 & 2 \end{vmatrix} R_2 \leftrightarrow R_1$$

$$= -\begin{vmatrix} 1 & 5 & 1 \\ 2 & 4 & 1 \\ 0 & -1 & 2 \end{vmatrix} R_2 \rightarrow -2R_1 + R_2$$

$$= \begin{vmatrix} 1 & 5 & 1 \\ 0 & -6 & -1 \\ 0 & -1 & 2 \end{vmatrix} R_2 \leftrightarrow R_3$$

$$= \begin{vmatrix} 1 & 5 & 1 \\ 0 & -1 & 2 \\ 0 & -6 & -1 \end{vmatrix} R_3 \rightarrow -6R_2 + R_3$$

$$= \begin{vmatrix} 1 & 5 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -13 \end{vmatrix}$$

$$= (1)(-1)(-13)$$

$$= 13$$

Example 2:

Find the determinant for matrix H by using PROPERTIES of determinant:

$$H = \begin{pmatrix} 4S+P & 4T+Q & 4U+R \\ -V & -W & -X \\ 2P & 2Q & 2R \end{pmatrix} \text{ given, } \begin{vmatrix} P & Q & R \\ S & T & U \\ V & W & X \end{vmatrix} = 12$$

Solution:

$$\begin{vmatrix} 4S+P & 4T+Q & 4U+R \\ -V & -W & -X \\ 2P & 2Q & 2R \end{vmatrix} = \begin{vmatrix} 4S & 4T & 4U \\ -V & -W & -X \\ 2P & 2Q & 2R \end{vmatrix} + \overset{\boxed{0}}{\begin{vmatrix} P & Q & R \\ -V & -W & -X \\ 2P & 2Q & 2R \end{vmatrix}}$$

$$=(-1)\ (4)\ (2) \begin{vmatrix} S & T & U \\ V & W & X \\ P & Q & R \end{vmatrix}$$

$$= (-)\ (-)\ (-8) \begin{vmatrix} P & Q & R \\ S & T & U \\ V & W & X \end{vmatrix} \Rightarrow \text{INTERCHANGE ROWS}$$

$$= (-8)(12)$$

$$= -96$$

Example 3:
Show that the determinant for the following matrix is equal to ZERO. Explain your answer WITHOUT using Laplace expansion.

$$\begin{vmatrix} 9 & 10 & 11 \\ 10 & 11 & 12 \\ 11 & 12 & 13 \end{vmatrix} \ \mathbf{= 0}$$

Solution:

$$\begin{vmatrix} 9 & 10 & 11 \\ 10 & 11 & 12 \\ 11 & 12 & 13 \end{vmatrix} R_3 \to -R_2 + R_3$$

$$= \begin{vmatrix} 9 & 10 & 11 \\ 10 & 11 & 12 \\ 1 & 1 & 1 \end{vmatrix} R_2 \to -R_1 + R_2$$

$$= \begin{vmatrix} 9 & 10 & 11 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \ R_2 = R_1 \text{ (identical)}$$

$$= 0$$

Example 4:

Calculate the determinant by using **properties**:

$$A = \begin{pmatrix} 1 & 4 & 4 \\ 2 & 3 & 2 \\ 2 & 5 & 7 \end{pmatrix}$$

Solution:

$$\begin{vmatrix} 1 & 4 & 4 \\ 2 & 3 & 2 \\ 2 & 5 & 7 \end{vmatrix} \begin{matrix} \\ R_2 \to -2R_1 + R_2 \\ R_3 \to -2R_1 + R_3 \end{matrix}$$

$$= \begin{vmatrix} 1 & 4 & 4 \\ 0 & -5 & -6 \\ 0 & -3 & -1 \end{vmatrix} R_2 \leftrightarrow R_3$$

$$= -\begin{vmatrix} 1 & 4 & 4 \\ 0 & -3 & -1 \\ 0 & -5 & -6 \end{vmatrix} C_2 \leftrightarrow C_3$$

$$= \begin{vmatrix} 1 & 4 & 4 \\ 0 & -1 & -3 \\ 0 & -6 & -5 \end{vmatrix} R_3 \to -6R_2 + R_3$$

$$= \begin{vmatrix} 1 & 4 & 4 \\ 0 & -1 & -3 \\ 0 & 0 & 13 \end{vmatrix}$$

$$= (1)(-1)(13)$$

$$= -13$$

Example 5:

Find the values of x if the determinant of $\begin{pmatrix} 5 & x & 3 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{pmatrix}$ is equal to zero.

(MUST USE PROPERTIES)

Solution:

$$\begin{vmatrix} 5 & x & 3 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} \begin{matrix} R_1 \to R_2 + R_1 \\ =0 \\ \\ \end{matrix}$$

$$\begin{vmatrix} x+7 & x+2 & 4 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} \begin{matrix} R_1 \to R_3 + R_1 \\ =0 \\ \\ \end{matrix}$$

$$\begin{vmatrix} x+4 & x+4 & x+4 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} = 0$$

$$(x+4)\begin{vmatrix} 1 & 1 & 1 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} = 0$$

$$(x+4)\begin{vmatrix} 1 & 1 & 1 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} \text{Transposition} \quad = 0$$

$$(x+4)\begin{vmatrix} 1 & x+2 & -3 \\ 1 & 2 & 2 \\ 1 & 1 & -x \end{vmatrix} R_3 \to -R_2 + R_3 \quad = 0$$

$$(x+4)\begin{vmatrix} 1 & x+2 & -3 \\ 1 & 2 & 2 \\ 0 & -1 & x-2 \end{vmatrix} R_2 \to -R_1 + R_2 \quad = 0$$

$$(x+4)\begin{vmatrix} 1 & x+2 & -3 \\ 0 & -x & 5 \\ 0 & -1 & x-2 \end{vmatrix} R_2 \leftrightarrow R_3 \quad = 0$$

$$(x+4)\begin{vmatrix} 1 & x+2 & -3 \\ 0 & -1 & x-2 \\ 0 & -x & 5 \end{vmatrix} R_3 \to -xR_2 + R_3 \quad = 0$$

$$(x+4)\begin{vmatrix} 1 & x+2 & -3 \\ 0 & -1 & x-2 \\ 0 & 0 & -x^2+2x+5 \end{vmatrix} R_3 \to -xR_2 + R_3 \quad = 0$$

$$(x+4)(1)(1)\left(-x^2+2x+5\right) = 0$$

$$(x+4)\left(-x^2+2x+5\right) = 0$$

$$x = -4\ ; \ x = 1 \pm \sqrt{6}$$

ADJOINT

Definition :
Let $\mathbf{A^T}$ be the transpose of an nxn matrix A. If every element of A^T is replaced by the corresponding **cofactor** respectively, then the resulting matrix is called the **adjoint of A** is denoted by **adj A**, i.e.

Recall that:

Cofactor

$$C_{ij} = \begin{pmatrix} +M_{11} & -M_{12} & +M_{13} & .. & (-1)^{1+j}M_{1j} & .. & (-1)^{1+n}M_{1n} \\ -M_{21} & +M_{22} & -M_{23} & .. & (-1)^{2+j}M_{2j} & .. & (-1)^{2+n}M_{2n} \\ +M_{31} & -M_{32} & +M_{33} & .. & (-1)^{3+j}M_{3j} & .. & (-1)^{3+n}M_{3n} \\ : & : & : & & : & & : \\ (-1)^{i+1}M_{i1} & (-1)^{i+2}M_{i2} & (-1)^{i+3}M_{i3} & .. & (-1)^{i+j}M_{ij} & .. & (-1)^{i+n}M_{1n} \\ : & : & : & & : & & : \\ (-1)^{n+1}M_{n1} & (-1)^{n+2}M_{n2} & .. & .. & (-1)^{n+j}M_{nj} & .. & (-1)^{n+n}M_{nn} \end{pmatrix}$$

Thus,

Adjoint of A= $(\mathbf{C_{ij}})^T$

$$\text{adj}(A) = \begin{pmatrix} +M_{11} & -M_{12} & +M_{13} & .. & (-1)^{1+j}M_{1j} & .. & (-1)^{1+n}M_{1n} \\ -M_{21} & +M_{22} & -M_{23} & .. & (-1)^{2+j}M_{2j} & .. & (-1)^{2+n}M_{2n} \\ +M_{31} & -M_{32} & +M_{33} & .. & (-1)^{3+j}M_{3j} & .. & (-1)^{3+n}M_{3n} \\ : & : & : & & : & & : \\ (-1)^{i+1}M_{i1} & (-1)^{i+2}M_{i2} & (-1)^{i+3}M_{i3} & .. & (-1)^{i+j}M_{ij} & .. & (-1)^{i+n}M_{1n} \\ : & : & : & & : & & : \\ (-1)^{n+1}M_{n1} & (-1)^{n+2}M_{n2} & .. & .. & (-1)^{n+j}M_{nj} & .. & (-1)^{n+n}M_{nn} \end{pmatrix}^T$$

Properties of adjoint:
If A and B are nxn matrices with $|A| \neq 0 and\ |B| \neq 0,$ then

i. det (adj A) $=|A|^{n-1}$ for n>2
ii. det(adjA)= $|A|^{n-1}A$ for n>2
iii. adj(A^{-1})=(adj A) $^{-1}$
iv. adj(A^T)=(adj A)^T
v. adj(AB)=(adj B)(adj A)

Example 1:

Show that Adj (A + B) = Adj(A) + Adj (B) , where A and B are 2x2 matrices,

$$A = \begin{bmatrix} 2 & 6 \\ 3 & 1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

Solution:

$$\text{Adj}(A) = \begin{bmatrix} 1 & -6 \\ -3 & 2 \end{bmatrix}$$

➔ For 2 x 2 : Interchange (a_{11}) ←→ (a_{22})

& change signs for a_{12} and a_{21}

$$\text{Adj}(B) = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

$$\text{Adj A + Adj B} = \begin{bmatrix} 1 & -6 \\ -3 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -8 \\ -2 & 3 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2 & 6 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 2 & 4 \end{bmatrix}$$

$$\text{Adj (A + B)} = \begin{bmatrix} 4 & -8 \\ -2 & 3 \end{bmatrix}$$

$$\therefore \text{Adj}(A + B) = \text{Adj}(A) + \text{Adj}(B)$$

Example 2:

If $A = \begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 0 & -2 \\ 4 & 1 & 5 \\ 0 & -3 & -1 \end{pmatrix}$, find adj (A)and adj(B).

Solution:

INVERSE OF A MATRIX

If A is a square matrix, a matrix B is called an ***inverse of A*** if and only if (iff) AB = I = BA. And we write $B = A^{-1}$, a matrix A that has inverse is called an ***invertible matrix***.

Definition:

An nxn matrix A is called *singular* if |A|=0

Theorem: Let A be a non-singular square matrix of order n, then $A^{-1} = \frac{1}{|A|}\text{adj}A$

Note that:

(i) If A is **non-singular**, i.e. $|A| \neq 0$, then $\mathbf{A^{-1}}$ **exist**.

(ii) If A is **singular**, i.e. $|A = 0$, then $\mathbf{A^{-1}}$ **does not exist**

Example 1:

Show that the matrix A is **NON-SINGULAR** and the determinant of A is equivalent to the determinant of A-transpose (A^T) :

Given $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 6 \\ -2 & 5 & 7 \end{pmatrix}$

Solution:

Non-singular ➔ $|A| \neq 0$

By using Laplace Expansion ➔ expands along R_1

$|A| = 1\begin{vmatrix} 1 & 6 \\ 5 & 7 \end{vmatrix} - 0\begin{vmatrix} 3 & 6 \\ -2 & 7 \end{vmatrix} + (-2)\begin{vmatrix} 3 & 1 \\ -2 & 5 \end{vmatrix}$

= (7 –30) - 0 + (-2)(15 + 2)

= -23 - 34

= -57 ➔ therefore matrix A is non-singular matrix.

$A^T = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & 6 & 7 \end{pmatrix}$

$|A^T| = 1\begin{vmatrix} 1 & 6 \\ 5 & 7 \end{vmatrix} - 0\begin{vmatrix} 3 & 6 \\ -2 & 7 \end{vmatrix} + (-2)\begin{vmatrix} 3 & 1 \\ -2 & 5 \end{vmatrix}$ Expands along the C_1

=- 57

Properties of the inverse matrix

1. $(A^T)^{-1} = (A^{-1})^T$
2. $(AB)^{-1} = B^{-1}A^{-1}, \quad (ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
3. *If A has an inverse, the inverse is unique.*

Example 1:

If $A=\begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}$ $B=\begin{pmatrix} 1 & 0 & -2 \\ 4 & 1 & 5 \\ 0 & -3 & -1 \end{pmatrix}$ find A^{-1} and B^{-1}.

Solution:

Example 2

If $A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & -4 \\ -1 & -1 & 3 \\ -1 & -2 & 5 \end{pmatrix}$, show that $B = A^{-1}$.

Solution:

Example 3

If $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, show that $A^3 = I$ and so find A^{-1}.

Solution:

Example 4:

If $A = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$, show that $A^2 - 3A - 10I = O$. Hence find A^{-1}.

Solution:

Example 5:
Find the inverse of the matrices M by using Adjoint Method.

$$M = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & 6 & 7 \end{pmatrix}$$

Solution:

$$M^{-1} = \frac{1}{|M|} \text{ Adj(M)}$$

$|M| = -57$

$$C_{ij} = \begin{pmatrix} \begin{vmatrix} 1 & 5 \\ 6 & 7 \end{vmatrix} & -\begin{vmatrix} 0 & 5 \\ -2 & 7 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -2 & 6 \end{vmatrix} \\ -\begin{vmatrix} 3 & -2 \\ 6 & 7 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ -2 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ -2 & 6 \end{vmatrix} \\ \begin{vmatrix} 3 & -2 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} \end{pmatrix}$$

$$\text{Adj (M)} = (\text{Cofactor M})^T = \begin{pmatrix} -23 & -10 & 2 \\ -33 & 3 & -12 \\ 17 & -5 & 1 \end{pmatrix}^T$$

$$= \begin{pmatrix} -23 & -33 & 17 \\ -10 & 3 & -5 \\ 2 & -12 & 1 \end{pmatrix}$$

$$M^{-1} = \frac{1}{|M|} \text{ Adj M}$$

$$= \frac{1}{-57} \begin{pmatrix} -23 & -33 & 17 \\ -10 & 3 & -5 \\ 2 & -12 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{23}{57} & \frac{11}{19} & \frac{-17}{57} \\ \frac{10}{57} & \frac{-1}{19} & \frac{5}{57} \\ \frac{-2}{57} & \frac{4}{19} & \frac{-1}{57} \end{pmatrix}$$

ELEMENTARY TRANSFORMATION OF MATRIX

The following operations are called elementary transformations.

1) $Ri \leftrightarrow Rj$: Interchange i th row and j th row
such that $R1 \leftrightarrow R2$: Interchange row 1 and row 2

2) $Ci \leftrightarrow Cj$: Interchange i th column and j th column
such that $C1 \leftrightarrow C2$: Interchange column 1 and column 2

3) kRi : Multiply each elements of i th row by a scalar k

4) kCi : Multiply each elements of j th column by a scalar k

5) Add to each elements of i^{th} row, k times the corresponding elements of j^{th} row: kRj + Ri

6) Add to each elements of j^{th} column, k times the corresponding elements of j^{th} row: kCj + Ci

<u>Equivalent Matrices</u>

Definition:
Two matrices A and B are said to be **equivalent**, denoted by $A \sim B$, if one can be obtained from the other by a sequence of elementary transformations.

<u>Note :</u>
Two matrices A and B are said to be **symmetric**, denoted by $A = A^T$,

Example 1:

$$A = \begin{pmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{pmatrix} -2R_1 + R_2$$

$$\sim \begin{pmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \end{pmatrix} 1R_1 + R_3$$

$$\sim \begin{pmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \end{pmatrix} -1R_2 + R_3$$

$\sim$ B

Finding the Inverse of a Matrix by the Elementary Row Operations (ERO)

Let A be a non-singular matrix so that A^{-1} exist. The following procedures can be used to find $A^{-1.}$

Procedures:
1. Write A and I side by side to form the augmented matrix (A:I)
2. Use ERO to transform A into I. The resulting matrix on the right hand side is the A^{-1}.

Example 1:
Find the inverse of A by using (i) Adjoint Method
(ii) Elementary Row Operation (ERO)

Let A be $\begin{pmatrix} 4 & -6 \\ -1 & 2 \end{pmatrix}$

Solution:

i) Adjoint Method :

$A^{-1} = \dfrac{1}{|A|}\text{Adj } A$

$|A| = 4(2) - (-6)(-1)$
$= 8 - 6$
$= 2$

$\text{Adj } A = \begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}$

$A^{-1} = \dfrac{1}{2}\begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 3 \\ \frac{1}{2} & 2 \end{pmatrix}$

ii) Elementary Row Operation :

$$\left(\begin{array}{cc|cc} 4 & -6 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right) R_2 \leftrightarrow R_1$$

$$\sim \left(\begin{array}{cc|cc} -1 & 2 & 0 & 1 \\ 4 & -6 & 1 & 0 \end{array}\right) R_2 \to 4R_1 + R_2$$

$$\sim \left(\begin{array}{cc|cc} -1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 4 \end{array}\right) R_2 \to \frac{1}{2}R_2$$

$$\sim \left(\begin{array}{cc|cc} -1 & 2 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & 2 \end{array}\right) R_1 \to -2R_2 + R_1$$

$$\sim \left(\begin{array}{cc|cc} -1 & 0 & -1 & -3 \\ 0 & 1 & \frac{1}{2} & 2 \end{array}\right) R_1 \to -R_1$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 3 \\ 0 & 1 & \frac{1}{2} & 2 \end{array}\right) \qquad \therefore A^{-1} = \begin{pmatrix} 1 & 3 \\ \frac{1}{2} & 2 \end{pmatrix}$$

Example 2:

Find the inverse of a matrix G if $G=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ by using ERO.

Solution:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \quad R_2 \to -2R_1 + R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -6 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 \to 2R_2 + R_1 \\ R_3 \to R_2 + R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -9 & -3 & 2 & 0 \\ 0 & -1 & -6 & -2 & 1 & 0 \\ 0 & 0 & -4 & -2 & 1 & 1 \end{array}\right) \quad R_3 \to -\frac{1}{4}R_3$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -9 & -3 & 2 & 0 \\ 0 & -1 & -6 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{-1}{4} & \frac{-1}{4} \end{array}\right) \quad \begin{array}{l} R_1 \to 9R_3 + R_1 \\ R_2 \to 6R_3 + R_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{4} & -\frac{9}{4} \\ 0 & -1 & 0 & 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{-1}{4} & \frac{-1}{4} \end{array}\right) \quad R_2 \to -R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{4} & -\frac{9}{4} \\ 0 & 1 & 0 & -1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{-1}{4} & \frac{-1}{4} \end{array}\right)$$

$$G^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{-1}{4} & \frac{-9}{4} \\ -1 & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix}$$

Example 3:

Find the inverse of the matrix A = $\begin{bmatrix} 4 & -1 & 2 \\ 5 & 0 & 0 \\ 6 & 1 & 3 \end{bmatrix}$, by using

(i) Elementary row operation
(ii) Adjoint method

Solution:

(i) Elemetary row operation

$$\left[\begin{array}{ccc|ccc} 4 & -1 & 2 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 \\ 6 & 1 & 3 & 0 & 0 & 1 \end{array}\right] R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 0 & 1 & 0 \\ 4 & -1 & 2 & 1 & 0 & 0 \\ 6 & 1 & 3 & 0 & 0 & 0 \end{array}\right] R_1 \rightarrow \frac{1}{5}R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{5} & 0 \\ 4 & -1 & 2 & 1 & 0 & 0 \\ 6 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 \rightarrow -4R_1 + R_2 \\ R_3 \rightarrow -6R_1 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & -1 & 2 & 1 & -\frac{4}{5} & 0 \\ 0 & 1 & 3 & 0 & -\frac{6}{5} & 1 \end{array}\right] R_3 \rightarrow R_2 + R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & -1 & 2 & 1 & -\frac{4}{5} & 0 \\ 0 & 0 & 5 & 1 & -2 & 1 \end{array}\right] R_3 \rightarrow \frac{1}{5}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & -1 & 2 & 1 & -\frac{4}{5} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & \frac{-2}{5} & \frac{1}{5} \end{array}\right] R_2 \rightarrow -2R_3 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & -1 & 0 & \frac{3}{5} & 0 & -\frac{2}{5} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{-2}{5} & \frac{1}{5} \end{array}\right] R_2 \rightarrow -R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 1 & 0 & -\frac{3}{5} & 0 & \frac{2}{5} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{-2}{5} & \frac{1}{5} \end{array}\right] \qquad \therefore A^{-1} = \begin{bmatrix} 0 & \frac{1}{5} & 0 \\ \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{1}{5} & \frac{-2}{5} & \frac{1}{5} \end{bmatrix}$$

(ii) Adjoint method

$$C_{ij} = \begin{bmatrix} 0 & -15 & 5 \\ 5 & 0 & -10 \\ 0 & 10 & 5 \end{bmatrix}$$

$$\text{Adj (A)} = \begin{bmatrix} 0 & 5 & 0 \\ -15 & 0 & 10 \\ 5 & -10 & 5 \end{bmatrix}$$

|A| = 25

$$A^{-1} = \frac{1}{|A|}\text{Adj } A$$

$$= \frac{1}{25}\begin{bmatrix} 0 & 5 & 0 \\ -15 & 0 & 10 \\ 5 & -10 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{5} & 0 \\ \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{1}{5} & \frac{-2}{5} & \frac{1}{5} \end{bmatrix}$$

SOLVING SYSTEM OF LINEAR EQUATIONS

A system of m linear equations in n unknowns has the following form:

$$a_{ss}x_x + a_{s2}x_{2x} + a_{s3}x_3 + ... + a_nx_n = b_1$$
$$a_{2s}x_x + a_{22}x_x + a_{23}x_3 + ... + a_{2s}x_x = b_2$$
:
:
$$a_{m1}x_x + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n = b_n$$

In matrix notation, the above system can be written as AX=B , where A is the coefficient matrix, x is the (unknown) column matrix, and B is the (constant) column matrix, i.e.

$$A = \begin{pmatrix} a11 & a12 & a13 & .. & a1j & .. & a1n \\ a21 & a22 & a23 & .. & a2j & .. & a2n \\ a31 & a32 & a33 & .. & a3j & .. & a3n \\ : & : & : & & : & & : \\ ai1 & ai2 & a13 & .. & aij & .. & a1n \\ : & : & : & & : & & : \\ an1 & an2 & .. & .. & anj & .. & ann \end{pmatrix}, \ X = \begin{pmatrix} x1 \\ x2 \\ x3 \\ : \\ : \\ bn \end{pmatrix} \text{ and } B = \begin{pmatrix} b1 \\ b2 \\ b3 \\ : \\ : \\ bm \end{pmatrix}$$

Definition:
The system is called **non-homogeneous** if B is not a zero matrix, i.e. at least one bi, is not zero for i=1,2,3,…,m

Definition:
The system is called **homogeneou**s if B is a zero matrix, i.e. at least one bi, is not zero for i =1,2,3,…,m

Definition:
A system of equation is **consistent** if it has at least a set of Solution::. Otherwise it is **inconsistent**.

Definition:
A system of m linear equations is n unknowns is said to be **dependent** if there exists k_1, k_2, k_3,…., k_m not all zero such that

$$k_1\ (a_{ss}x_x + a_{s2}x_{2x} + a_{s3}x_3 + ... + a_n x_n - b_1)$$
$$+\quad k_2\ (a_{2s}x_x + a_{22}x_x + a_{23}x_3 + ... + a_{2s}x_x - b_2)$$
$$:+ \dots .. \quad + \quad k_m\ (a_{m1}x_x + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n - b_n) = 0$$

Otherwise, the system is considered **independent**.

Method 1: By matrix Inversion

The system in matrix form : **AX = B**

Assuming that A is non-singular, i.e. A^{-1} exists, we have

$$.\ A^{-1}.(AX) = A^{-1}.B$$
$$(A^{-1}.\ A)X = A^{-1}.B$$
$$IX = A^{-1}.\ B$$

Thus, $\boxed{\mathbf{X = A^{-1}B}}$

Note : This method is only suitable when m = n and cannot be used if A is singular.

Example 1:
Solve the following system of equations

$$2x_1 + 4x_2 + 6x_3 = 18$$
$$4x_1 + 5x_2 + 6x_3 = 24$$
$$2x_1 + x_2 - 2x_3 = 4$$

by finding the inverse of the coefficient matrix (inversion method).

Solution:
Let,

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 2 & 1 & -2 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 18 \\ 24 \\ 4 \end{bmatrix} \quad , \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Where $X = A^{-1}B$ and $A^{-1} = \frac{1}{|A|}\text{adj}(A)$

$|A| = 12$

$$C_{ij} = \begin{bmatrix} -16 & 20 & -6 \\ 14 & -16 & 6 \\ -6 & 12 & -6 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} -16 & 14 & -6 \\ 20 & -16 & 12 \\ -6 & 6 & -6 \end{bmatrix}$$

$$X = \frac{1}{12}\begin{bmatrix} -16 & 14 & -6 \\ 20 & -16 & 12 \\ -6 & 6 & -6 \end{bmatrix}\begin{bmatrix} 18 \\ 24 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Example 2:

Find the values of x, y and z for the following system of equation by matrix inversion

$$2x + y = 1$$
$$-y + 3z = 7$$
$$3x - z = 1$$

Solution:

Method II : Cramer's Rule

This method is suitable for the system of n linear equations in n unknowns : **AX= B**

$$\text{Let } A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & .. & a_{1j} & .. & a_{1n} \\ a_{21} & a_{22} & a_{23} & .. & a_{2j} & .. & a_{2n} \\ a_{31} & a_{32} & a_{33} & .. & a_{3j} & .. & a_{3n} \\ : & : & : & & : & & : \\ a_{i1} & a_{i2} & a_{13} & .. & a_{ij} & .. & a_{in} \\ : & : & : & & : & & : \\ a_{n1} & a_{n2} & .. & .. & a_{nj} & .. & a_{nn} \end{pmatrix} \text{ and } D = |A|$$

Also, let $|A_i| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & .. & a_{1j} & .. & a_{1n} \\ a_{21} & a_{22} & a_{23} & .. & a_{2j} & .. & a_{2n} \\ a_{31} & a_{32} & a_{33} & .. & a_{3j} & .. & a_{3n} \\ : & : & : & & : & & : \\ a_{i1} & a_{i2} & a_{13} & .. & a_{ij} & .. & a_{in} \\ : & : & : & & : & & : \\ a_{n1} & a_{n2} & .. & .. & a_{nj} & .. & a_{nn} \end{vmatrix}$

Such that, by replacing i^{th} column of |A| by the elements in B for i = 1, 2, 3, . …, n

The Cramer's Rule : $x_i = \dfrac{|A_i|}{|A|}, \text{ for } i = 1,2,3...,n$

Conclusions:

Under the method of Cramer's Rule :

- i- If $|A| \neq 0$,the system of equations is **consistent** and has a set of solution (is **uniqu**e).
- ii- If $|A| = 0$ but $|A_i| \neq 0$, for some i=1,2,3,…,n the system of equations is **inconsistent** and **no solution**
- iii- If $|A| = 0$ but $|A_i| = 0$, for all i-1,2,3,…,n the system of equations is **consistent** and has **infinitely many solutions (IMS)** .

Example 1:
Solve the simultaneous linear equations by using Cramer's Rule:

$3x_1 - x_2 + 4x_3 = -2$
$x_1 + 2x_2 - x_3 = -3$
$-2x_1 + 3x_2 + x_3 = 5$

Solution:

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -2 \\ -3 \\ 5 \end{bmatrix}$$

$|A| =$

$|A_x| =$

$|A_y| =$

$|A_z| =$

$$x = \frac{|A_X|}{|A|} = \frac{-84}{42} = -2$$

$$y = \frac{|A_y|}{|A|} = \frac{0}{42} = 0$$

$$z = \frac{|A_z|}{|A|} = \frac{42}{42} = 1$$

Try to find x, y and z by using Inversion method:

$$X = \frac{1}{|A|} \text{Adj } A \bullet B$$

Method III: Solving by using Gauss Elimination Method
(Homogeneous and Non-Homogenous Equations)

Row Echelon Form

***Definition*:**
A matrix is said to be in **row echelon form** (REF) if it satisfies the following three conditions:

i- If there are all zeros rows, then it must appear at the bottom of the matrix.

ii- The first non-zero number must appear from the right in each non-zero row (called as **a pivot) ,** and if the pivot is equal to '1' then it is called **'the leading 1'.**

iii- Each leading 1 is to the right of the leading 1 in the preceding row.

If in addition, the matrix satisfies one more condition, which is

iv- The column containing the leading 1 has zeroes elsewhere,
then the matrix is said to be in **reduced row echelon form** (RREF)
or (Canonical form).

Example 1:
Some of the **row echelon form (REF)** matrices :

1) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ 2) $\begin{pmatrix} -1 & 2 & 3 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{pmatrix}$ 3) $\begin{pmatrix} 1 & -5 & 0 & -1 \\ 0 & 2 & 3 & 9 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ 4) $\begin{pmatrix} 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}$

Some of the **reduced row echelon form (RREF**) matrices :

1) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 2) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ 3) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ 4) $\begin{pmatrix} 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Example 2:
Using elementary row-operations reduce the following matrices to row echelon form (REF) and reduced row echelon form (RREF).

a) $\begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & -2 & 1 \\ 1 & 3 & -2 & 0 \\ 2 & 4 & - & 4 \\ 1 & 1 & -1 & 6 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 1 & 3 & 1 & 0 & 1 \\ -2 & 3 & 0 & 4 & 2 & -2 \\ 0 & 1 & 1 & 0 & -1 & -4 \end{pmatrix}$

Solution:

Rank of a matrix

Definition:
The number of **non-zero rows** of a matrix A in row echelon form (or reduced row echelon form) is called the rank of A and is devoted by rank A or r(A).

Example 1:

1) Augmented matrix of A = $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, thus rank of A = r(A) = 2

2) Augmented matrix of B = $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, thus rank of A = r(A) = 1

Example 2:
Find the rank of the following matrices:

a) $\begin{pmatrix} 2 & -1 & 1 \\ 4 & 5 & 0 \\ -2 & 0 & 4 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 0 & -2 \\ 1 & 2 & 0 \\ 3 & 1 & 5 \\ 0 & 4 & 8 \end{pmatrix}$

Solution:

Solving a systems of NON-HOMOGENEOUS linear equations

The system of m linear equations in n unknowns can be solved by applying the elementary row operations:

Procedures :

i. Write the equations AX=B in the form of augmented matrix (A| B)
ii. Use the elementary row operations to reduce the augmented matrix into row echelon form.
iii. Rewrite the equation from the reduced form and do the **'backward substitution method'** in order to find the values of the unknowns.

Rouche Theorem:

The system of linear equations AX=B is **consistent** if and only if (iff) the rank of matrix A is equal to the rank of the augmented matrix (A|B) i.e. r(A)=r(A|B).

Theorem:

If the augmented matrix (A|B) in the row echelon form or reduced row echelon form possesses a zero row, the system of linear equations is **linearly dependent**.

Three types of Solution::s may occur:

1- r(A) $\neq$ (A|B) , or
r > n where n=number of column in A,
or [0000…0 | *] ,
The system of equations is **inconsistent** and has **no Solution::**.

2- r(A) =(A|B)=n , or **r = n**
The system of equations is **consisten**t and has **unique Solution::**

3- r(A) =(A|B) = **r < n**
The system of equations is **consisten**t and has **infinitely many solutions (IMS).**
In this case , r unknowns can be written as the linear combination of (n-r) unknowns to which may be assigned arbitrary values (parameters).

Example 1:

Solve the following system of **non-homogeneous** equations. Show that the answers is in an infinitely many solutions (IMS). Clearly show its steps, rank, dependency and consistency.

Given that

$$x_1+3x_2-2x_3+5x_4=4$$
$$x_1+4x_2+x_3+3x_4=5$$
$$x_1+4x_2+2x_3+4x_4=3$$
$$2x_1+7x_2-3x_3+6x_4=13$$

Solution:

$$A=\begin{pmatrix}1&3&-2&5\\1&4&1&3\\1&4&2&4\\2&7&-3&6\end{pmatrix}\qquad B=\begin{pmatrix}4\\5\\3\\13\end{pmatrix}\qquad X=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$$

$$\left(\begin{array}{cccc|c}1&3&-2&5&4\\1&4&1&3&5\\1&4&2&4&3\\2&7&-3&6&13\end{array}\right)\begin{array}{l}R_2\to -R_1+R_2\\R_3\to -R_1+R_3\\R_4\to -2R_1+R_4\end{array}$$

$$\left(\begin{array}{cccc|c}1&3&-2&5&4\\0&1&3&-2&1\\0&1&4&-1&-1\\0&1&1&-4&5\end{array}\right)\begin{array}{l}R_3\to -R_2+R_3\\R_4\to -R_2+R_4\end{array}$$

$$\left(\begin{array}{cccc|c}1&3&-2&5&4\\0&1&3&-2&1\\0&0&1&1&-2\\0&0&-2&-2&4\end{array}\right)R_4\to 2R_3+R_4$$

$$\left(\begin{array}{cccc|c}1&3&-2&5&4\\0&1&3&-2&1\\0&0&1&1&-2\\0&0&0&0&0\end{array}\right)$$

⇨ Row 4 :
all zeros , therefore it is an linearly <u>DEPENDENT</u>.

⇨ Rank = r = 3 and Column = n = 4
Therefore, $r < n$ ➔ <u>CONSISTENT</u> & <u>INFINITELY MANY SOLUTION:</u>.

Backward substitution:

R_4: -

R_3: $x_3+x_4=-2$

$x_3=-2-x_4$ ➔ Let , $x_4=t$

$x_3=-2-t$

R_2: $x_2+3x_3-2x_4=1$

$x_2=1-3x_3+2x_4$

$=1-3(-2-t)+2t$

$=7+5t$

R_1: $x_1+3x_2-2x_3+5x_4=4$

$x_1=4-3x_3+2x_3-5x_4$

$=4-3(7+5t)+2(-2-t)-5t$

$=-21-22t$

Example 2:
Show that the non-homogeneous system is INCONSISTENT.
Given that

$$\begin{aligned} x - y + z + w &= 1 \\ 2x - 2y - 3z - 3w &= 17 \\ -x + y + 2z + 2w &= -10 \\ x - y - z - w &= 9 \end{aligned}$$

Solution:

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 1 \\ 2 & -2 & -3 & -3 & 17 \\ -1 & 1 & 2 & 2 & -10 \\ 1 & -1 & -1 & -1 & 9 \end{array}\right) \begin{array}{l} R_2 \to 2R_1 + R_2 \\ R_3 \to R_1 + R_3 \\ R_4 \to -R_1 + R_4 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & -5 & -5 & 15 \\ 0 & 0 & 3 & 3 & -9 \\ 0 & 0 & -2 & -2 & 8 \end{array}\right) R_2 \to -\frac{1}{5}R_2$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 3 & 3 & -9 \\ 0 & 0 & -2 & -2 & 8 \end{array}\right) \begin{array}{l} R_3 \to -3R_2 + R_3 \\ R_4 \to 2R_2 + R_4 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right) \quad R_3 \leftrightarrow R_4$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Since row 3 is in the form of **[0 0 0 … 0 | b]**, thus this is an **inconsistent** linear system. Therefore, no Solution.

COMPARISONS AMONG THE THREE METHODS : Inversion Method , Cramer's Rule and Gauss Elimination method

Example 3:

Given : $4x + 4y - 8 = 0$
$x + 2y = 8$

Find the values of x and y.

Solve the above system of **non-homogeneous** linear equations by using:

(i) Inversion Method
(ii) Cramer's Rule
(iii) Gauss Elimination

Solution:

Let : $A = \begin{pmatrix} 4 & 4 \\ 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 8 \\ 8 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \end{pmatrix}$

(i) **Using Matrix Inversion** : ➔ apply only for square matrix (m x m)

$$X = A^{-1} B$$
$$= \left(\frac{1}{|A|} \text{Adj } A\right)(B)$$

➔ $|A| = 4(2) - 4$
$= 4$ ➔ non-singular.

$$\text{Adj } A = \begin{pmatrix} 2 & -4 \\ -1 & 4 \end{pmatrix}$$

➔ $X = \frac{1}{4}\begin{pmatrix} 2 & -4 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 8 \\ 8 \end{pmatrix}$

$$= \frac{1}{4}\begin{pmatrix} 2(8) + (-4)(8) \\ (-1)(8) + (4)(8) \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -16 \\ 24 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$$

(ii) **Cramer's Rule :**

$|A| = 4$

$|A_x| = \begin{vmatrix} 8 & 4 \\ 8 & 2 \end{vmatrix} = 16\text{-}32 = \text{-}16$

$|A_y| = \begin{vmatrix} 2 & 8 \\ -1 & 8 \end{vmatrix} = 16 + 8 = 24$

$x = \frac{|A_x|}{|A|} = \frac{-16}{4} = -4$

$y = \frac{|A_y|}{|A|} = \frac{24}{4} = 6$

(iii) **Gauss Elimination :**

$$\left(\begin{array}{cc|c} 4 & 4 & 8 \\ 1 & 2 & 8 \end{array}\right) R_1 \to \frac{1}{4} R_1$$

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 2 & 8 \end{array}\right) R_2 \to -R_1 + R_2$$

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 6 \end{array}\right)$$ ➔ consistent (r=n=2) & unique

Backward substitution:

R_2 ➔ y = 6 #

R_1 ➔ x + y = 2

x= 2-y

= 2 –6

= - 4

Example 4:
Solve for x, y, and z , for the following **non-homogeneous** equations ,
by using :

(i) Cramer's Rule
(ii) Gauss Elimination
state its dependency, consistency and rank.

Given : 2x + 4y = 5
4x – 2y + z = -1
5y + x – 2z = 3

Solution:

$$A=\begin{pmatrix} 2 & 4 & 0 \\ 4 & -2 & 1 \\ 1 & 5 & -2 \end{pmatrix} \qquad B=\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} \qquad X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

|A|= 2 (4 – 5) – 4 (-8 –1) ➔ expands along row 1
=2(-1)- 4 (-9)
=-2 + 36
=34

Cramer's Rule :

|A|=34

$$|A_x|=\begin{vmatrix} 5 & 4 & 0 \\ -1 & -2 & 1 \\ 3 & 5 & -2 \end{vmatrix} =5(-1)-4(-1)=-1$$

$$|A_y|=\begin{vmatrix} 2 & 5 & 0 \\ 4 & -1 & 1 \\ 1 & 3 & -2 \end{vmatrix} = -2 + 45 = 43$$

$$|A_z|=\begin{vmatrix} 2 & 4 & 5 \\ 4 & -2 & -1 \\ 1 & 5 & 3 \end{vmatrix} = 2(-1) -4(13) +5(22)= 56$$

$$x=\frac{|A_x|}{|A|}=\frac{-1}{34}$$

$$y=\frac{|A_y|}{|A|}=\frac{43}{34}$$

$$z=\frac{|A_z|}{|A|}=\frac{28}{17}$$

Gauss Elimination :

$$\left(\begin{array}{ccc|c} 2 & 4 & 0 & 5 \\ 4 & -2 & 1 & -1 \\ 1 & 5 & -2 & 3 \end{array}\right) \begin{array}{l} R_2 \to -2R_1 + R_2 \\ R_3 \to -\frac{1}{2}R_1 + R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 4 & 0 & 5 \\ 0 & -10 & 1 & -11 \\ 0 & 3 & -2 & \frac{1}{2} \end{array}\right) R_2 \to -\frac{1}{10}R_2$$

$$\left(\begin{array}{ccc|c} 2 & 4 & 0 & 5 \\ 0 & 1 & -\frac{1}{10} & \frac{11}{10} \\ 0 & 3 & -2 & \frac{1}{2} \end{array}\right) R_3 \to -3R_2 + R_3$$

$$\left(\begin{array}{ccc|c} 2 & 4 & 0 & 5 \\ 0 & 1 & -\frac{1}{10} & \frac{11}{10} \\ 0 & 0 & -\frac{17}{10} & -\frac{28}{10} \end{array}\right)$$

➔ **rank (r) =number of column(n) = 3 / consistent/ unique/ independent**

Backward substitution :

R_3 : $\frac{-17}{10}z = -\frac{28}{10}$

$$z=\frac{28}{17}$$

R_2: $y-\frac{1}{10}z=\frac{11}{10}$

$$=\frac{11}{10}+\frac{1}{10}\left(\frac{28}{17}\right)$$

$$=\frac{43}{34}$$

R_1: $2x + 4y = 5$

$$2x = 5-4\left(\frac{43}{34}\right)$$

$$=-\frac{1}{17}$$

$$x=\frac{-1}{34}$$

Example 5:
Find the values of t such that the system **of non-homogeneous** equations are :
(i) inconsistent
(ii) having infinitely many solution
(iii) having unique solution.

Given that :
$$tx + 3y - z = 11$$
$$x + 2y + z = 2$$
$$-tx + y + 2z = -1$$

Solution:

$$\left(\begin{array}{ccc|c} t & 3 & -1 & 11 \\ 1 & 2 & 1 & 2 \\ -t & 1 & 2 & -1 \end{array}\right) R_3 \to R_1 + R_3$$

$$\left(\begin{array}{ccc|c} t & 3 & -1 & 11 \\ 1 & 2 & 1 & 2 \\ 0 & 4 & 1 & 10 \end{array}\right) R_1 \leftrightarrow R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ t & 3 & -1 & 0 \\ 0 & 4 & 1 & 10 \end{array}\right) R_2 \to -tR_1 + R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 3-2t & -1-t & 1-2t \\ 0 & 4 & 1 & 10 \end{array}\right) R_3 \leftrightarrow R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 4 & 1 & 10 \\ 0 & 3-2t & -1-t & 11-2t \end{array}\right) R_2 \to \frac{1}{4}R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & \frac{1}{4} & \frac{5}{2} \\ 0 & 3-2t & -1-t & 11-2t \end{array}\right) R_3 \to (2t-3)R_2 + R_3$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & \frac{1}{4} & \frac{5}{2} \\ 0 & 0 & -\frac{7}{4}-\frac{t}{2} & 3t+\frac{7}{2} \end{array}\right)$$

Based on **ROUCHE THEOREM** :
(i) **INCONSISTENT :** If number of rank is not equal to the number of column ➔ r ≠ n ≠3 ;

$$\frac{-7}{4} - \frac{t}{2} = 0 \text{ but } 3t + \frac{7}{2} \neq 0$$

$$t = -\frac{7}{2} \text{ but } t \neq -\frac{7}{6}$$

$$\therefore t = -\frac{7}{2}$$

(ii) **INFINITELY MANY SOLUTION:** If the number of rank is less than the number of columns ➔ $r < n$;

$$\frac{-7}{4} - \frac{t}{2} = 0 \text{ and } 3t + \frac{7}{2} = 0$$

$$t = -\frac{7}{2} \text{ and } t = -\frac{7}{6}$$

$$\therefore \textbf{impossible}$$

(iii) **UNIQUE SOLUTION:** If the number of rank is equivalent to the number of columns ➔ $r = n = 3$;

$$\frac{-7}{4} - \frac{t}{2} \neq 0$$

$$t \neq -\frac{7}{2}$$

Solving a system of HOMOGENEOUS linear equations:

The system of homogeneous equations : $AX=0$

The augmented matrix : $(A \mid 0)$

The system is always consistent since $r(A) = r(A \mid 0)$

Rouche Theorem (homogeneous):

There are THREE possibilities to be considered:

1. If $r(A)=n$=number of unknowns, then the system of equalities of equations has a ***TRIVIAL*** Solution::, $x_i = 0$, for all $i=1,2,3,\ldots,n$

2. If $r(A) = r < n$, then the system of homogeneous equations has a ***NON-TRIVIAL*** Solution::s (infinitely many solution::s) where r unknowns can be expressed as a linear combination of (n-r) unknowns to which arbitrary values (parameters) may be assigned.

3. If $r(A) = r > n$, then the system of homogeneous equations is considered inconclusive.

Example 1:
Show that the linear system has a TRIVIAL Solution:: (HOMOGENEOUS EQUATIONS).
Given that

$$\begin{aligned} x + y - z &= 0 \\ 2x - 4y + 3z &= 0 \\ 3x + 7y - z &= 0 \end{aligned}$$

Solution:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -4 & 3 & 0 \\ 3 & 7 & -1 & 0 \end{array}\right) \quad \begin{array}{l} R_2 \to -2R_1 + R_2 \\ R_3 \to -3R_1 + R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -6 & 5 & 0 \\ 0 & 4 & 2 & 0 \end{array}\right) \quad R_2 \to -\frac{1}{6}R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{5}{6} & 0 \\ 0 & 4 & 2 & 0 \end{array}\right) \quad R_3 \to -4R_2 + R_3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{5}{6} & 0 \\ 0 & 0 & \frac{16}{3} & 0 \end{array}\right) \Rightarrow \boxed{\text{Rank}(r) = \text{column}(n) = 3}$$

Therefore,
CONSISTENT AND TRIVIAL SOLUTION➔ $\boxed{x = y = z = 0}$

Example 2:
Show that the solutions for this homogeneous equation is infinitely many solution (IMS) and **Non-trivial .**

$$\begin{aligned} x + y - 2z + 3w + 2k &= 0 \\ 2x - y + 3z + 4w + k &= 0 \\ -x - 2y + 2z + w &= 0 \\ 3x + z + 7w + 3k &= 0 \end{aligned}$$

Solution:

$$\left(\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 2 & 0 \\ 2 & -1 & 3 & 4 & 1 & 0 \\ -1 & -2 & 3 & 1 & 0 & 0 \\ 3 & 0 & 1 & 7 & 3 & 0 \end{array}\right) \begin{array}{l} \\ R_2 \to -2R_1 + R_2 \\ R_3 \to R_1 + R_3 \\ R_4 \to -3R_1 + R_4 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 2 & 0 \\ 0 & -3 & 7 & -2 & -3 & 0 \\ 0 & -1 & 1 & 4 & 2 & 0 \\ 0 & -3 & 7 & -2 & -3 & 0 \end{array}\right) R_2 \leftrightarrow R_3$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 2 & 0 \\ 0 & -1 & 1 & 4 & 2 & 0 \\ 0 & -3 & 7 & -2 & -3 & 0 \\ 0 & -3 & 7 & -2 & -3 & 0 \end{array}\right) \begin{array}{l} \\ \\ R_3 \to -3R_2 + R_3 \\ R_4 \to -3R_2 + R_4 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 2 & 0 \\ 0 & -1 & 1 & 4 & 2 & 0 \\ 0 & 0 & 4 & -14 & -9 & 0 \\ 0 & 0 & 4 & -14 & -9 & 0 \end{array}\right) \begin{array}{l} \\ \\ \\ R_4 \to -R_3 + R_4 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 2 & 0 \\ 0 & -1 & 1 & 4 & 2 & 0 \\ 0 & 0 & 4 & -14 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Backward substitution :

R_4: -

R_3: $4z - 14w - 9k = 0$

$$4z = 14w + 9k$$

$$4z = 14p + 9q \qquad \rightarrow \text{let} \quad \mathbf{w = p \text{ and } k = q}$$

$$z = \frac{7}{2}p + \frac{9}{4}q$$

R_2: $-y + z + 4w + 2k = 0$

$$y = z + 4w + 2k$$

$$y = \left(\frac{7}{2}p + \frac{9}{4}q\right) + 4p + 2q$$

$$y = \frac{15}{2}p + \frac{17}{4}q$$

R_1: $x + y - 2z + 3w + 2k = 0$

$$x = -y + 2z - 3w - 2k$$

$$x = -\left(\frac{15}{2}p + \frac{17}{4}q\right) + 2\left(\frac{7}{2}p + \frac{9}{4}q\right) - 3p - 2q$$

$$x = -\frac{7}{2}p - \frac{7}{4}q$$

Example 3:

Solve the following system of homogeneous linear equations:

$$x - y + 2z + w = 0$$

$$3x + 2y + w = 0$$

$$4x + y + 2z + 2w = 0$$

Solution:

Example 4:

Find the values of k for which the system

$$kx + y + z = 0$$
$$x + y - z = 0$$
$$x + y + kz = 0$$

has **nontrivial solutions** and determine the solutions in each case.

Solution:

EIGENVALUES, EIGENVECTORS AND NORMAL EIGENVECTORS

The characteristic equation is the equation which is solved to find a **matrix's eigenvalues**, also called the **characteristic polynomial**. For a general n x n matrix A, the **characteristic equation** in variable λ or k is defined by

$$|A - \lambda I| = 0 \quad \text{or } |A - kI| = 0$$

where I is the identity matrix and A is matrix coefficient:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

so the **characteristic equation** is given by

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & & . \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0$$

The solution λ of the characteristic equation are called **eigenvalues** and are extremely important in the analysis of many problems in mathematics and physics. The polynomial left-hand side of the characteristic equation is known as the **characteristic polynomial.**

Theorem:

The eigenvector, X of A corresponding to an eigenvalues k is **<u>not unique.</u>**

Definition: The magnitude of a eigenvector, X= $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ be denoted by |x| is defined by

$$|X| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2}$$

Definition

If X is an eigenvector corresponding to the eigenvalue k or λ then the unit vector $\vec{u} = \frac{x}{|x|}$ is called the **<u>normalized eigenvector</u>**.

Example 1:

Find the eigenvalues, eigenvector and normal eigenvectors of the following matrices.

$$A=\begin{pmatrix}4 & 6\\ 8 & 2\end{pmatrix} \qquad B=\begin{pmatrix}x\\ y\end{pmatrix}$$

Solution:

Characteristic equation ➔ $|A - kI| = 0$

$$\left|\begin{pmatrix}4 & 6\\ 8 & 2\end{pmatrix} - k\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right| = 0$$

$$\begin{vmatrix}4-k & 6\\ 8 & 2-k\end{vmatrix} = 0$$

(4-k)(2-k) –6(8) = 0

$8-4k-2k+k^2-48=0$

$k^2-6k-40=0$

(k – 10)(k + 4)=0

➔ $\boxed{\textbf{k= 10 or -4}}$ ⇨ **EIGENVALUES**

case 1 :

LET k = 10 ;

$$(A - 10I)X = 0 \text{ where } X=\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{bmatrix}4-10 & 6\\ 8 & 2-10\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Using Gauss Elimination:

$$\left[\begin{array}{cc|c}-6 & 6 & 0\\ 8 & -8 & 0\end{array}\right] R_1 \to \frac{1}{6}R_1$$

$$\left[\begin{array}{cc|c}-1 & 1 & 0\\ 8 & -8 & 0\end{array}\right] R_2 \to 8R_1 + R_2$$

$$\left[\begin{array}{cc|c}-1 & 1 & 0\\ 0 & 0 & 0\end{array}\right]$$

Backward substitution :

R_2: ---

R_1: -x + y = 0

x = y ➔ **let** $\boxed{\textbf{y = t}}$

➔ x = t

EIGENVECTOR FOR k=10 ➔ $\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}t\\ t\end{pmatrix} = t\begin{pmatrix}1\\ 1\end{pmatrix}$

Magnitude of X ➔ **|X|=** $\sqrt{t^2+t^2}$

=$t\sqrt{2}$

NORMAL EIGENVECTOR (k=10) :

$$U= \frac{X}{|X|}$$

$$=\frac{1}{t\sqrt{2}} \bullet t\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$$

case 2 :
LET k = - 4 ;

$$(A+4I)X = 0 \text{ where } X=\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{bmatrix} 4+4 & 6 \\ 8 & 2+4 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using Gauss Elimination:

$$\left[\begin{array}{cc|c} 8 & 6 & 0 \\ 8 & 6 & 0 \end{array}\right] R_2 \to -R_1 + R_2$$

$$\left[\begin{array}{cc|c} 8 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Backward substitution :

R_2: ---

R_1: $8x + 6y = 0$

$x = -\frac{3}{4}y$ ➔ **let** $\boxed{y = t}$

➔ $x = -\frac{3}{4}t$

EIGENVECTOR FOR k=-4 ➔ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{3}{4}t \\ t \end{pmatrix} = t\begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix}$

Magnitude of X ➔ **|X|=** $\sqrt{(-\frac{3}{4}t)^2 + t^2}$

= $\frac{5}{4}t$

NORMAL EIGENVECTOR (k=-4) :

$$U = \frac{X}{|X|}$$

$$= \frac{4}{5t} \bullet t \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

Example 2:

Show that the eigenvalues for the following system are 1 , $\frac{3}{4}$ and $\frac{1}{2}$.

Given that : A = $\begin{pmatrix} 1 & \frac{1}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$ X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Find the eigenvector and normal eigenvectors for the **<u>LARGEST eigenvalue</u>** from the following matrices.

Solution:

Characteristic equation ➔ | A – kI | = 0

$$\begin{vmatrix} 1-k & \frac{1}{4} & 0 \\ 0 & \frac{3}{4}-k & \frac{1}{2} \\ 0 & 0 & \frac{1}{2}-k \end{vmatrix} = 0$$

use Laplace expansion : (1-k) [($\frac{3}{4}$ - k) ($\frac{1}{2}$ - k)] = 0

Factorize : 1-k = 0 ; $\frac{3}{4}$ - k = 0 ; $\frac{1}{2}$ - k = 0

$\boxed{\mathbf{k = 1}}$ $\boxed{\mathbf{k} = \frac{3}{4}}$ $\boxed{\mathbf{k} = \frac{1}{2}}$

SHOWN THAT THE EIGENVALUES ARE 1, $\frac{1}{2}$ AND $\frac{3}{4}$

<u>LARGEST eigenvalue</u> ➔ LET k = 1

$$\left(\begin{array}{ccc|c} 1-1 & \frac{1}{4} & 0 & 0 \\ 0 & \frac{3}{4}-1 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2}-1 & 0 \end{array}\right) = 0$$

$$\left(\begin{array}{ccc|c} 0 & \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \end{array}\right) R_2 \to R_1 + R_2$$

$$\left(\begin{array}{ccc|c} 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & -1/2 & 0 \end{array}\right) R_3 \to R_2 + R_3$$

$$\left(\begin{array}{ccc|c} 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

BACKWARD SUBSTITUTION

R_3: ---

R_2: $\frac{1}{2}z = 0$ ➔ z=0

R_1: $\frac{1}{4}y = 0$ ➔ y=0

SINCE variable x are not defined , we just can't assume that x is equal to zero. Instead, we should assume that x is equal to other variable such that **x = t** .

Eigenvector for k= 1 ➔ $X = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Magnitude for X ➔ $|X| = \sqrt{t^2}$

Normal Eigenvector for k= 1 ➔ $U = \frac{X}{|X|}$

$$= \frac{1}{t}\begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

* TRY solving for **k =** $\frac{3}{4}$ **and k =** $\frac{1}{2}$

Example 3:
Show that the eigenvalues for the following matrix is 1, 2 and 2 . Then, calculate the Normal Eigenvector corresponding to the SMALLEST eigenvalue.

Given that: $A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{pmatrix}$; $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Solution:

Characteristic equation ➔ $|\ A - kI\ | = 0$

$$\begin{vmatrix} 4-k & 6 & 6 \\ 1 & 3-k & 2 \\ -1 & -5 & -2-k \end{vmatrix} = 0$$

ERO ➔ $$\begin{vmatrix} 4-k & 6 & 6 \\ 0 & -2-k & -k \\ -1 & -5 & -2-k \end{vmatrix} = 0$$

Use Laplace Expansion ➔ $(4\text{-}k)\ [\ (2+k)^2 - 5k\ ] - [-6k+12+6k] = 0$

$$4k^2 - 4k + 16 - k^3 + k - 4k - 12 = 0$$

$$k^3 - 5k^2 + 8k - 4 = 0$$

$$(k-1)(k-2)^2 = 0$$

➔ $\mathbf{k = 1, 2, 2}$

NOTE

$$\begin{array}{r l} & k^2 - 4k + 4 \\ (k\text{-}1) & \overline{)k^3 - 5k^2 + 8k - 4} \\ & \underline{-(k^3 - k^2)} \\ & -4k^2 + 8k \\ & \underline{-(-4k^2 + 4k)} \\ & 4k - 4 \\ & \underline{-(4k-4)} \\ & - \quad - \end{array}$$

LET **k =1** , ➔ smallest eigenvalue

$$\left(\begin{array}{ccc|c} 4-1 & 6 & 6 & 0 \\ 1 & 3-1 & 2 & 0 \\ -1 & -5 & -2-1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 3 & 6 & 6 & 0 \\ 1 & 2 & 2 & 0 \\ -1 & -5 & -3 & 0 \end{array}\right) R_1 \to \frac{1}{3}R_1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 1 & 2 & 2 & 0 \\ -1 & -5 & -3 & 0 \end{array}\right) \begin{array}{l} \\ R_2 \to -R_1 + R_2 \\ R_3 \to \quad R_1 + R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -1 & 0 \end{array}\right) R_2 \leftrightarrow R_3$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Backward substitution,

R_3: ---

R_2: $-3y - z = 0$ ➔ **Let $z = t$**

$$y = -\frac{1}{3}z$$

Thus, $y = -\frac{1}{3}t$

R_1: $x + 2y + 2z = 0$

$$x = -2y - 2z$$

$$= -2\left(-\frac{1}{3}\right)t - 2t$$

$$= -\frac{4}{3}t$$

Corresponding <u>Eigenvector</u> ➔ $\mathbf{X} = \begin{pmatrix} \frac{4}{-3}t \\ \frac{1}{-3}t \\ t \end{pmatrix} = t\begin{pmatrix} \frac{4}{-3} \\ \frac{1}{-3} \\ 1 \end{pmatrix}$

Magnitude ➔ $\mathbf{|X|} = \sqrt{(-\frac{4}{3}t)^2 + (-\frac{1}{3}t)^2 + (t)^2}$

$$= \sqrt{\frac{26}{9}t^2}$$

$$= \frac{\sqrt{26}}{3}t$$

Normal Eigenvector ➔ $\mathbf{U} = \frac{X}{|X|} = \left(\frac{3}{\sqrt{26}t}\right)t\begin{pmatrix} \frac{4}{-3} \\ \frac{1}{-3} \\ 1 \end{pmatrix}$

$$= \frac{1}{\sqrt{26}} \cdot \begin{pmatrix} -4 \\ -1 \\ 3 \end{pmatrix}$$ #

Example 4:

Find the eigenvalues and normalized eigenvector corresponding to the LARGEST eigenvalue for the following matrix

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

Solution:

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Eric W. Weisstein. "Characteristic Equation." From *MathWorld*--A Wolfram Web Resource.

Exercise 1.1 (Matrix Operation)

1. Given that E = $\begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{4} & \mathbf{-7} \\ \mathbf{-2} & \mathbf{6} \end{bmatrix}$, F = $\begin{bmatrix} \mathbf{-3} & \mathbf{6} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}$ and G = $\begin{bmatrix} \mathbf{1} & \mathbf{6} \\ \mathbf{-7} & \mathbf{-1} \\ \mathbf{-4} & \mathbf{3} \end{bmatrix}$. Find EF – 4G.

Answers:

$$EF - 4G = \begin{bmatrix} -11 & -3 \\ 2 & 7 \\ 34 & -6 \end{bmatrix}$$

2. Find x, y, w and z if

$$\mathbf{3}\begin{bmatrix} \mathbf{x} & \mathbf{y} \\ \mathbf{z} & \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{x} & \mathbf{6} \\ \mathbf{-1} & \mathbf{2w} \end{bmatrix} + \begin{bmatrix} \mathbf{4} & \mathbf{x+y} \\ \mathbf{z+w} & \mathbf{3} \end{bmatrix}$$

Answers:

(x,y,z,w) = (2,4,1,3)

3. If $\mathbf{P} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{2} \end{bmatrix}$

(i) Evaluate that $P^2 - 3P + I$
(ii) By using the result from (i), find P^{-1}

Answers:

i. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

ii. $\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

4. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$, evaluate A^3 and A^4.

Then, deduce the matrices A^{40} and A^{49}.

Answers:

$A^3 = A^{49}$ & $A^4 = A^{40}$

Exercise 1.2 (Finding inverse using elementary row operation)

1. Given $A = \begin{bmatrix} 1 & -1 & 4 \\ 2 & 1 & -1 \\ 3 & 2 & -1 \end{bmatrix}$,

Find A^{-1} by using elementary row operations.

Answers:

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & 7 & -3 \\ -1 & -13 & 9 \\ 1 & 5 & 3 \end{bmatrix}$$

2. Identify whether the following matrix B is singular or non-singular

$$\begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & 2 \end{bmatrix}.$$

Find B^{-1} using elementary row operation.

Answers:

$|B| \neq 0$ ➔ non-singular matrix

$$B^{-1} = \frac{1}{-41}\begin{bmatrix} -20 & -2 & 3 \\ 19 & 6 & -9 \\ 12 & -7 & -10 \end{bmatrix}$$

3. Use the elementary row operation to find the inverse of the following matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$$

Answers:

$$A^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

Exercise 1.3 (Determinant)

1.Prove that,

$$\begin{vmatrix} \mathbf{x^2+y^2} & \mathbf{y^2+z^2} \\ \mathbf{xy} & \mathbf{zy} \end{vmatrix} = \mathbf{y(z-x)(y^2-xz)}$$

by using the properties of the determinant.

2. Given that

$$A = \begin{bmatrix} \mathbf{t+3} & \mathbf{-1} & \mathbf{1} \\ \mathbf{5} & \mathbf{t-3} & \mathbf{1} \\ \mathbf{6} & \mathbf{-6} & \mathbf{t+4} \end{bmatrix}$$

Express the determinant of matrix A as a product of three linear factors.

Hence, determine the values of **t** for which the matrix A is a singular matrix.

Answers:

t = -2, 2, -4;
singular matrix ➔ |A| = 0.

3. Given that $\begin{vmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \end{vmatrix}$ = 7, use the properties of determinant to evaluate the following determinant.

$$\begin{vmatrix} \mathbf{u+x} & \mathbf{-3x} & \mathbf{4a} \\ \mathbf{v+y} & \mathbf{-3y} & \mathbf{4b} \\ \mathbf{w+z} & \mathbf{-3z} & \mathbf{4c} \end{vmatrix}.$$

Answers:
84

4. Given that $\mathbf{A} = \begin{bmatrix} \mathbf{x} & \mathbf{0} & \mathbf{2} \\ \mathbf{1} & \mathbf{-1} & \mathbf{0} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{2} & \mathbf{x} \\ \mathbf{0} & \mathbf{-1} \end{bmatrix}$. Find x such that det(AB) = det(BA).

Answers:
x = 2

5. Calculate $\begin{vmatrix} \mathbf{-3k} & \mathbf{-3l} & \mathbf{-3m} \\ \mathbf{p-2x} & \mathbf{q-2y} & \mathbf{r-2z} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \end{vmatrix}$ by using properties of determinant, if

$\begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{p} & \mathbf{q} & \mathbf{r} \\ \mathbf{k} & \mathbf{l} & \mathbf{m} \end{vmatrix} = \mathbf{4}$.

Answers:

12

6. If $\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} & \mathbf{i} \end{vmatrix}$ = 10. Evaluate the following determinant.

(i) $\begin{vmatrix} \mathbf{-a} & \mathbf{b} & \mathbf{c} \\ \mathbf{-5d} & \mathbf{5e} & \mathbf{5f} \\ -\frac{\mathbf{1}}{\mathbf{3}}\mathbf{g} & \frac{\mathbf{1}}{\mathbf{3}}\mathbf{h} & \frac{\mathbf{1}}{\mathbf{3}}\mathbf{i} \end{vmatrix}$

(ii) $-5\begin{vmatrix} \mathbf{g} & \mathbf{h} & \mathbf{i} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{a+2g} & \mathbf{b+2h} & \mathbf{c+2i} \end{vmatrix}$

Answers:

i. –50 / 3

ii. –50

Exercise 1.4 (Finding inverse using Adjoint Method)

1. Use the Adjoint method to calculate the inverse of matrix $A = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{bmatrix}$

Answers:

$$A^{-1} = (1/6)\begin{bmatrix} -16 & 14 & -6 \\ 26 & -22 & 12 \\ -11 & 10 & -6 \end{bmatrix}$$

2. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$ by using the Adjoint Method.

Answers:

$$A^{-1} = 1/6\begin{bmatrix} 1 & 3 & -5 \\ 1 & 3 & 1 \\ 1 & -3 & 1 \end{bmatrix}$$

Exercise 1.5 (Cramer's Rule)

1.By using the Cramer's rule, find the values of k that will make the following system having

(i) a unique solution,
(ii) an infinitely many solutions.
(iii) inconsistent

$kx + 2y = 1$
$2x + (k-3)y = -2$

Answers:

i. unique solution if $D \neq 0 \Rightarrow k \neq 4$ or $k \neq -1$
ii. infinitely many solution $\Rightarrow D = 0$ & $D_x=0 \Rightarrow k=-1$
iii. $D=0$ but $D_{xi} \neq 0 \Rightarrow k=4$

2. (i) Solve the following non- homogeneous system of equation using Cramer's rule

$\mathbf{a}X + 2Y - 3Z = \mathbf{b}$
$2X - Y + 4Z = 2$
$4X + 3Y - 2Z = 14$

where a and b are constants.

(ii) What are the values of **a** and **b,** that will make the system an inconsistent system.

Answers:

i. $x = \dfrac{b-6}{a-1}$; $y = \dfrac{6a-2b+6}{a-1}$; $z = \dfrac{2a+4-b}{a-1}$

ii. inconsistent $\Rightarrow$ no sol. $\Rightarrow D = 0$; $a = 1$, b can be any value

3. Given

$$y = 2x + b(x - a)$$
$$z = x + b(y - a) + 2y$$
$$y = 2z + b(z - a)$$

By using Cramer's rule, show that $y = \dfrac{ab}{2+b}$.

4. Consider the following system of linear equations in x, y and z

$x - 2y + 4z = 1$
$\mathbf{m}x + 2y + \mathbf{n}z = 1$
$-2x + 4y + z = -1$

where m and n are constants. Use Cramer's rule to:

(i) solve for z

(ii) find the values of m and n so that the system has a unique solution.

Answers:

i. $z = 1/9$

ii. unique sol. If $D \neq 0$, then $m \neq -1$

Exercise 1.6 (Inversion Method)

1. Given that $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

(i) By using the adjoint method, find the inverse of matrix A.

(ii) Use the above result in solving the following system of equation for the unknowns x, y, and z.

$$x + 2y + 3z = 5$$
$$2x + 5y + 3z = 3$$
$$x + 8z = 17$$

Answers:

(i) $A^{-1} = \frac{1}{|A|} \text{adj } A$

$$= -1 \begin{bmatrix} -40 & -16 & -9 \\ -13 & 5 & 3 \\ -5 & 2 & 1 \end{bmatrix}$$

(ii) $(x, y, z) = (1, -1, 2)$

2. Find the inverse of the matrix A using the adjoint method, given that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$$

Hence, solve the following simultaneous equations using the above result.

2y + 3z = 2 – x
2x + 3y = -4z – 3
x + 7z = 8 – 5y

Answers:

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & -3 & -1 \end{bmatrix}$$

thus, x = -9/2 ; y = -8; z = 15/2

3. Use the elementary row operations method to find the inverse of matrix

$$M = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 5 & -4 \\ 1 & -4 & 6 \end{bmatrix}.$$

Hence, solve the following system of linear equations using the above result.

x – 2y + z = 1
-2x + 5y – 4z = 2
x – 4y + 6z = 1

Answers:

$$M^{-1} = \begin{bmatrix} 14 & 8 & 3 \\ 8 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 & 8 & 3 \\ 8 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 33 \\ 20 \\ 8 \end{bmatrix}$$

4. Given the following system of linear equation in variables x , y and z

$$(p-1)x + 3y - 3z = 5$$
$$-3x + (p+5)y - 3z = 7$$
$$-6x + 6y + (p-4)z = -1$$

where **p** is a constant.

i) Write the above system in matrix form: AX = B.

ii) Find the value of **p** that will make matrix A a non-singular matrix.

iii) Find the inverse of A when p = 2.

iv) Solve the system of equation when p = 2.

Answers:

i. $$\begin{bmatrix} p-1 & 3 & -3 \\ -3 & p+5 & -3 \\ -6 & 6 & p-4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ -1 \end{bmatrix}$$

ii. $|A| = (p+2)^2 (p-4)$
& to be non-singular : $p \neq -2$ or $p \neq 4$

iii. $$A^{-1} = -\frac{1}{32}\begin{bmatrix} 4 & -12 & 12 \\ 12 & -20 & 12 \\ 24 & -24 & 6 \end{bmatrix}$$

iv. $x = 19/8$; $y = 23/8$; $z = 2$

Exercise 1.7 (Gauss Elimination Method)

1. Use Gaussian elimination method to reduce following system to row echelon form and, discuss the consistency and dependency the system.

$$\begin{aligned} -X_2 - X_3 + X_4 &= 0 \\ X_1 + X_2 + X_3 + X_4 &= 6 \\ 2X_1 + 4X_2 + X_3 - 4X_4 &= -1 \\ 3X_1 + X_2 - 2X_3 + 2X_4 &= 3 \end{aligned}$$

Hence, solve the above system for X_1, X_2, X_3, and X_4.

Answers:

$r = 4 = n$ ➔ consistent – unique sol. & independent ➔ no zero row

$(X_1, X_2, X_3, X_4) = (10, -9, 7, -2)$

2. Reduce the matrix $N = \begin{bmatrix} 1 & -2 & 1 & 5 \\ -1 & -3 & 2 & 2 \\ 2 & 1 & -1 & 3 \\ 3 & 4 & -3 & 1 \end{bmatrix}$ to row echelon form and identify the rank(N).

Hence, solve the following system of linear equations for t, u, v, and w.

t –2u + v + 5w = 0
-t – 3u + 2v + 2w = 0
2t + u – v + 3w = 0
3t + 4u – 3v + w = 0

Answers:

rank (N) = 2
v = r & w = s then t = (1/5)(r-(1/s))
u = (1/5) (3r + 7s)
v = r & w = s.

3. Use Gauss elimination method to reduce the following system of homogeneous equation to row echelon form and states the consistency and dependency of the system.

$$\begin{bmatrix} 2 & 2 & 2 \\ -2 & 5 & 2 \\ -7 & 7 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, solve the system for x, y and z.

Answers:

consistent & dependent

let z = t, then $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{3}{7}t \\ -\frac{4}{7}t \\ t \end{bmatrix}$

4. Solve the following system of linear equations by using Gauss elimination method

$$x + 2y + 3z = 3$$
$$2x + y + z - 3w = 9$$
$$3x + 2y + 3z - 3w = 10$$

State the consistency and dependency of the system.

Answers:

Independent (no zero row) ; Consistent .
z = t then, x = -1 + t , y = 2- 3t , z = t & w = -3

5. Consider the following system of simultaneous equations in the unknown variables x, y, and z.

$$x + 2y - z = -3$$
$$3x + 5y + \mathbf{k}z = -4$$
$$9x + (\mathbf{k} + 13)y + 6z = 9 \qquad \text{where } \mathbf{k} \text{ is a constant.}$$

(i) Show that the augmented matrix of this system can be reduced by row operation to the following form:

$$\begin{bmatrix} 1 & 2 & -1 & : & -3 \\ 0 & 1 & -k-3 & : & -5 \\ 0 & 0 & k^2-2k & : & 5k+11 \end{bmatrix}$$

(ii) Use the result of part (i) to solve the following system of equations

$x + 2y - z = -3$
$3x + 5y - z = -4$
$9x + 12y + 6z = 9$

(iii) For what values of k does the original system not have a solution?

Answers:

i. show = $\left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & 1 & -k-3 & -5 \\ 0 & 0 & k^2-2k & 5k+11 \end{array}\right]$

ii. $x = 1$, $y = -1$ & $z = 2$

iii. $k = 0, 2$

6. Determine the values of **a** so that the following system in unknown x, y, and z has

(i) infinitely many solutions
(ii) a unique solution.

$x + y - z = 1$
$2x + 3y + \mathbf{a}z = 3$
$x + \mathbf{a}y + 3z = 2$

Give reason for each of the answer above.

Answers:

i infinitely many solutions $r < n$ ➔ a=2
ii. unique solution : $r = n$ ➔ $a \neq -3, 2$

7. For the following system of linear equations with unknowns x, y and z

$2x + 3y + 2z = 5$
$x + y + z = 2$
$2x + 3y + (m^2 - 7)z = m + 2$

a) determine the values of m so that the system :
   (i) is linearly dependent
   (ii) is consistent
   (iii) has a unique solution.

b) Find the solution for part (a) when m = 5.

Answers:

a) i. linearly dependent if m =3
   ii. inconsistent if m = -3
   iii. unique solution if m $\neq \pm 3$

(b) if m =5, z = 1/8
   y = 1
   x = 7/8

8. Given the following system of homogeneous equation

$$x + y - z = 0$$
$$2x + 3y - \alpha z = 0$$
$$x + \alpha y + 3z = 0$$

Find the values of α that will make the system having non-trivial solutions and determine the solutions in each case.

Answers:

α = 2, -3 : when $\alpha = 2$, ➔ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5t \\ -4t \\ t \end{bmatrix}$

$\alpha = -3$ ➔ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$

Exercise 1.8 (Finding inverse using elementary row operation)

1. Given

$$A = \begin{bmatrix} 1 & 6 & 0 \\ 2 & 2 & 0 \\ 0 & 3 & 4 \end{bmatrix}$$

Find the eigenvector of matrix A corresponding to the largest eigenvalue of A.

Answers:

eigenvalue = k = 5

$$\text{eigenvector} = x_I = \begin{bmatrix} t/2 \\ t/3 \\ t \end{bmatrix} = t\begin{bmatrix} 1/2 \\ 1/3 \\ 1 \end{bmatrix}$$

2. Given k =1 is one of the eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & -12 & 4 \\ -1 & 0 & -2 \\ -1 & 5 & -1 \end{bmatrix}$$

(i) find the other eigenvalues

(ii) find the eigenvector corresponding to the given eigenvalue.

Answers:

i. char. Eq. = $-k^3 + 2k^2 + k - 2$; eigenvalues = -1, 1, 2

ii. $\text{eigenvector} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

3. Given that $B = \begin{bmatrix} 2 & -1 & 0 \\ -6 & 2 & 1 \\ 2 & -3 & 2 \end{bmatrix}$

(i) Show that the characteristic equation of B is $-\lambda^3 + 6\lambda^2 - 9\lambda = 0$. Hence, find all the eigenvalues of B.

(ii) Find the normal eigenvector corresponding to the smallest eigenvalue of B.

Answers:

i. eigenvalues = 0 , 3 ➔ smallest ev =0

ii.eigenvector = $\begin{bmatrix} t/2 \\ t \\ t \end{bmatrix}$; Normal eigenvectors = $(1/3)\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

4. Determine the characteristic polynomial and eigenvalues of matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -2 & 0 \\ 5 & 1 & 2 \end{bmatrix}.$$

Hence, find the normal eigenvector corresponding to the smallest eigenvalue.

Answers:

eigenvalue = 2,3,-2

eigenvector : $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -4t \\ t \end{bmatrix}$

normal eigenvector = $\frac{1}{\sqrt{17}}\begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix}$

5. Write down the characteristic equation and determine the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$$

Find the eigenvector corresponding to the smallest eigenvalue.

Answers:

eigenvector ~ smallest eigenvalue => $x = \begin{bmatrix} -t \\ -4t \\ t \end{bmatrix}$

Chapter Two

VECTOR ALGEBRA

CHAPTER 2 : VECTOR ALGEBRA

INTRODUCTION:
A **vector** is a quantity with **magnitude** and a **direction**.
examples: velocity, acceleration, force and magnetic field intensities.

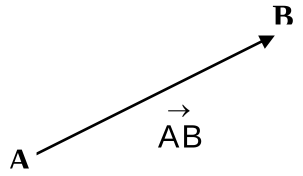
A **scalar** is a real number. Scalar quantities posses magnitude **but not** direction.
examples: mass, volume, temperature, density, work, etc.

Example 1:
State which of the following are vector quantities and which are scalars.

i. weight
ii. volume
iii. density
iv. momentum
v. speed
vi. distance
vii. electric field

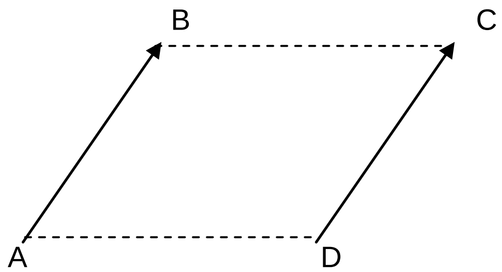
Answer :
i,iv,vii (vectors) ii,iii,v,vi (scalars)

VECTORS:
A vector is represented by a directed line segment. The direction of the vector is indicated by an arrow pointing from the initial point to the terminal point.
If the initial point is at A and the terminal point is at B, the vector from A to B is written $\overrightarrow{AB}$. The length (magnitude) of a vector is written $\left|\overrightarrow{AB}\right|$. The length is always **a non-negative real number**.



VECTOR OPERATIONS:

a. Equality Of Vectors



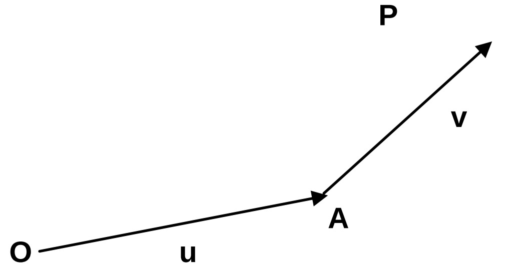
Two vectors are equal if and only if they have the same magnitude and direction. For example, if the line segment AB and DC have the same direction and the same length, then ABCD is a parallelogram and the vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are equal;
we write this as $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{BC} = \overrightarrow{AD}$.

b. Addition Of Vectors

There are two procedures for addition of vectors; the triangle law and the parallelogram law .

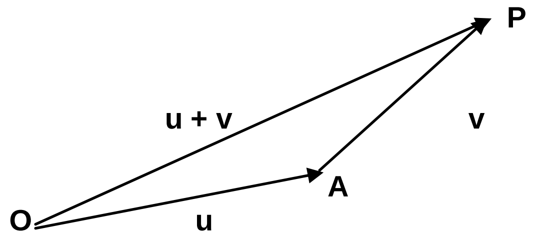
i. Triangle Law

Place the initial point of the vector **v** at the terminal point of the vector **u**. Let **u** = $\overrightarrow{OA}$ and **v** = $\overrightarrow{AP}$.



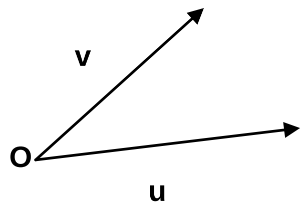
Now construct the vector $\overrightarrow{OP}$ to complete the third side of the triangle OAP.

The vector **u + v** defined to be the vector $\overrightarrow{OP}$.



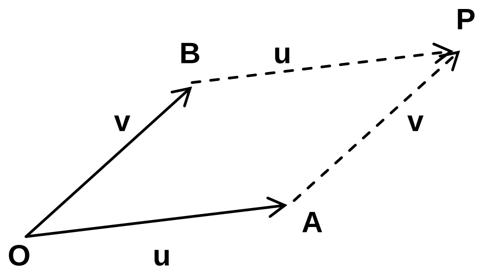
ii. Parallelogram Law

Suppose **u** and **v** are two vectors. Place them so that they are with a common initial point O.



From the terminal point of each vectors, draw a copy of the other vector to complete a parallelogram OAPB. In this parallelogram, u = $\overrightarrow{OA} = \overrightarrow{BP}$ and v = $\overrightarrow{OB} = \overrightarrow{AP}$

The vector u + v is defined to be the vector $\overrightarrow{OP}$



c. Vector arithmetic

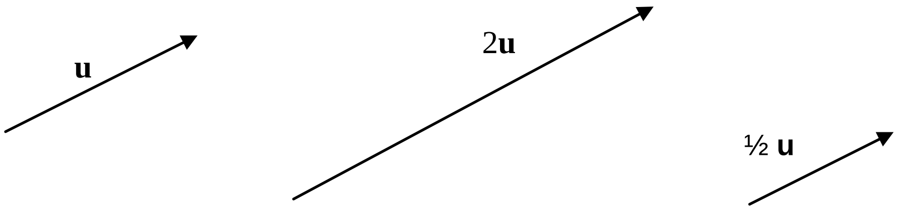
Theorem: Let **u, v** and **w** be vectors and *k, l* be scalars. Then

i.	**u** + **v** = **v** + **u**	(commutative)
ii.	(**u** + **v**) + **w** = **u** + (**v** + **w**)	(associative)
iii.	**u** + **0** = **u**,	where **0** = (0,0), or 0= (0,0,0)
iv.	**u** + (**– u**) = 0	where **– u** = (–1)**u**
v.	*k* (*l* **u**) = (*kl*) **u**	(distributive law)
vi.	*k*(**u** + **v**) = *k***u** +*k***v**, (*k* – *l*)**u** = k**u** + l**u**	(distributive law)
vii.	1**u** = **u**	

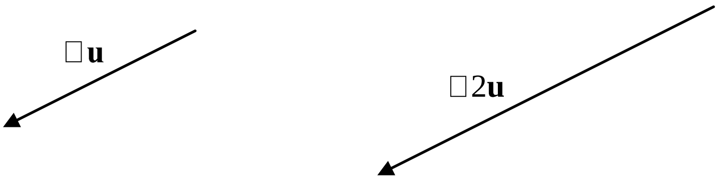
d. Scalar Multiplication : *k* **u**

If k is a scalar and **u** is a vector, then the scalar multiple k**u** is the vector whose length is $|k|$ times the length of **u.**

k > 0 same direction



k < 0 opposite direction



Exercises:

1. For the vector $\vec{a} = \langle 2,4 \rangle$ compute $3\vec{a}$, $\frac{1}{2}\vec{a}$ and $-2\vec{a}$.

2. Determine if the sets of vectors are parallel or not.

   (a) $\vec{a} = \langle 2,-4 \rangle$, $\vec{b} = \langle -6,12 \rangle$

   (b) $\vec{a} = \langle 4,10 \rangle$, $\vec{b} = \langle 2,-9 \rangle$

VECTORS IN TWO DIMENSIONS:

Vector Components and Vector unit $\vec{i}$ **and** $\vec{j}$

Let the initial point of a vector **v** at the origin of a rectangular coordinate system , then its terminal point of **v** has coordinates of the form (a_1, a_2)
These coordinates are called the components of **v** and we write

$$\mathbf{v} = \langle a_1, a_2 \rangle$$

The notation $\langle a_1, a_2 \rangle$ for the ordered pair that refers to a vector so as not to confuse it with ordered pair (a_1, a_2) that refers to a point in the plane.

Let $i = \langle 1, 0 \rangle$ and $j = \langle 0, 1 \rangle$, then **i** and **j** are vectors that have **length 1** and point in the direction of the positive x- and y- axes.

If $\mathbf{v} = \langle a_1, a_2 \rangle$, then we can write

$$\begin{aligned} \mathbf{v} &= \langle a_1, a_2 \rangle \\ &= \langle a_1, 0 \rangle + \langle 0, a_2 \rangle \\ &= a_1 \langle 1, 0 \rangle + a_2 \langle 0, 1 \rangle \\ &= a_1 \mathbf{i} + a_2 \mathbf{j} \end{aligned}$$

Thus, any vector in V_2 can be expressed in terms of the standard basis vectors **i** and **j**.

Example 1:
$\langle 1, -4 \rangle = \mathbf{i} - 4\mathbf{j}$

VECTOR OPERATIONS:

a. Equality of vectors: $a_1 \vec{i} + b_1 \vec{j} = a_2 \vec{i} + b_2 \vec{j} \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$

b. Vector addition and subtraction :

Let $\vec{v}_1 = a_1 \vec{i} + b_1 \vec{j}$ and $\vec{v}_2 = a_2 \vec{i} + b_2 \vec{j}$ then

i. $\vec{v}_1 + \vec{v}_2 = \left(a_1 \vec{i} + b_1 \vec{j}\right) + \left(a_2 \vec{i} + b_2 \vec{j}\right) = (a_1 + a_2)\vec{i} + (b_1 + b_2)\vec{j}$

ii. $\vec{v}_1 - \vec{v}_2 = \left(a_1 \vec{i} + b_1 \vec{j}\right) - \left(a_2 \vec{i} + b_2 \vec{j}\right) = (a_1 - a_2)\vec{i} + (b_1 - b_2)\vec{j}$

c. Length(magnitude) of a vector : If $\vec{v} = a\vec{i} + b\vec{j}$ **,** $\left|\vec{v}\right| = \sqrt{a^2 + b^2}$

d. Multiplications of scalars:

$c\vec{v} = c\left(a\vec{i} + b\vec{j}\right) = (ca)\vec{i} + (cb)\vec{j}$, where c is a scalar, and $\left|c\vec{v}\right| = |c|\left|\vec{v}\right|$

e. Zero vector: The vector $\vec{0} = 0\vec{i} + 0\vec{j}$ is called the zero vector. It is only vector whose length is zero.

Example 1:

Find $|\vec{a}|$, $\vec{a}+\vec{b}$, $\vec{a}-\vec{b}$, $2\vec{a}$, *and* $3\vec{a}+2\vec{b}$.

i. $\mathbf{a}=\langle 2,4\rangle$, $\mathbf{b}=\langle 3,-1\rangle$ ii. $\mathbf{a}=\mathbf{i}+2\mathbf{j}$, $\mathbf{b}=3\mathbf{i}-\mathbf{j}$

Solution:

i.
$$|\mathbf{a}|=\sqrt{2^2+4^2}=\sqrt{4+16}=\sqrt{20}$$
$$\mathbf{a}+\mathbf{b}=\langle 2+3.4+(-1)\rangle=\langle 5,3\rangle$$
$$\mathbf{a}-\mathbf{b}=\langle 2-3,4-(-1)\rangle=\langle -1,5\rangle$$
$$2\mathbf{a}=2\langle 2,4\rangle=\langle 4,8\rangle$$
$$\begin{aligned}3\mathbf{a}+2\mathbf{b}&=3\langle 2,4\rangle+2\langle 3,-1\rangle\\&=\langle 6,12\rangle+\langle 6,-2\rangle\\&=\langle 6+6,12+(-2)\rangle\\&=\langle 12,10\rangle\end{aligned}$$

ii.
$$|\mathbf{a}|=\sqrt{1^2+2^2}=\sqrt{1+4}=\sqrt{5}$$
$$\mathbf{a}+\mathbf{b}=(1+3)\vec{i}+(2+(-1))\vec{j}=4\vec{i}+\vec{j}$$
$$\mathbf{a}-\mathbf{b}=(1-3)\vec{i}+(2-(-1))\vec{j}=-2\vec{i}+3\vec{j}$$
$$2\mathbf{a}=2(\vec{i}+2\vec{j})=2\vec{i}+4\vec{j}$$
$$\begin{aligned}3\mathbf{a}+2\mathbf{b}&=3(\vec{i}+2\vec{j})+2(3\vec{i}-\vec{j})\\&=(3\vec{i}+6\vec{j})+(6\vec{i}-2\vec{j})\\&=9\vec{i}+4\vec{j}\end{aligned}$$

Exercises

1. Vectors **v** and **u** are given by their components as follows; **v**=<-2,3> and **u** = <4,6>. Find each of the following vectors.

   a. **v** + 2**u**
   b. **u** – 4**v**

2. Vectors **v** and **u** are given by **v** = <4 , 1> and **u** = <**u1** , **u2**>, find components **u1** and **u2** so that 2**v** - 3**u** $=\vec{0}$.

UNIT VECTOR AND DIRECTION OF VECTORS

A unit vector is a vector whose length is 1. For instance $\vec{i}$ and $\vec{j}$ are unit vectors. If **a** (position vector) $\neq 0$,then the unit vector that has the same direction as **a** is

$$\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

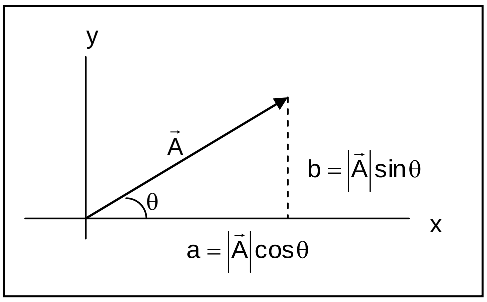
Proving:

For any vector , $\vec{A} = a\,\vec{i} + b\,\vec{j}$,

$$\vec{A} = \left(\left|\vec{A}\right|\cos\theta\right)\vec{i} + \left(\left|\vec{A}\right|\sin\theta\right)\vec{j}$$

$$= \left|\vec{A}\right|\left(\vec{i}\cos\theta + \vec{j}\sin\theta\right)$$

$$= \sqrt{a^2 + b^2}\left(\vec{i}\cos\theta + \vec{j}\sin\theta\right)$$

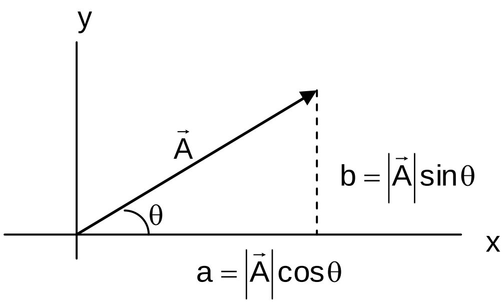


$$\vec{i}\cos\theta + \vec{j}\sin\theta = \frac{\vec{A}}{\sqrt{a^2 + b^2}}$$

$$\vec{u} = \frac{\vec{A}}{\sqrt{a^2 + b^2}}$$

Note: where $\square\pi < \theta \leq \pi$, $\vec{u} = \vec{i}\cos\theta + \vec{j}\sin\theta$

Angle between Vector and Axis:



$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{b}{a}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Example 1:

Find a unit vector in the same direction as $\mathbf{a} = \langle 3, -4\rangle$.

Solution:

$$|\mathbf{a}| = |\langle 3, -4\rangle| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$$

Direction of $\mathbf{u} = \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{1}{5}\langle 3, -4\rangle = \left\langle \dfrac{3}{5}, -\dfrac{4}{5}\right\rangle$

Example 2:

Let $\begin{matrix} \vec{u} = 2\lambda \vec{i} + \lambda \vec{j} \\ \vec{v} = \lambda \vec{i} - 2\vec{j} \end{matrix}$,

For λ = 2, find the unit vector in the direction of $2\vec{u} - \vec{v}$

Solution:

Example 3:

Given $\vec{E} = 5\sqrt{3}\ \vec{i} - 5\ \vec{j}$. Find the unit vector in the direction of $\vec{E}$ and find the angle made with the positive x-axis.

Solution:

The given vector has length

$$\left|\vec{E}\right| = \sqrt{(5\sqrt{3})^2 + (-5)^2} = \sqrt{25(3) + 25} = \sqrt{100} = 10$$

the unit vector with the same direction is

$$\hat{E} = \frac{\left\langle 5\sqrt{3}, -5 \right\rangle}{10} = \frac{5\sqrt{3}\ \vec{i} - 5\ \vec{j}}{10}$$

$$\tan\theta = \frac{-5}{5\sqrt{3}} = \frac{-1}{\sqrt{3}}$$

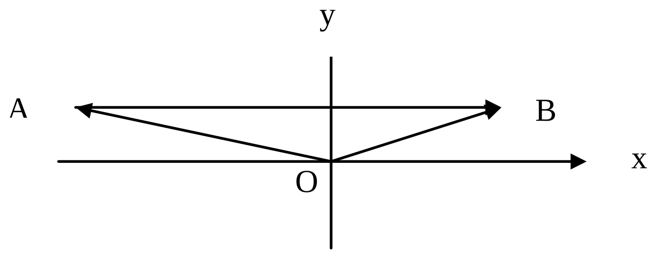
$$\theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = 330^\circ$$

Exercises:

1. Find a unit vector that points in the same direction as $\vec{w} = \langle -5, 2 \rangle$
2. Given that **u** = $\sqrt{3}$ **i**- 2**j**, find the unit vector in the opposite direction to **u**

POSITION VECTOR

If P (p_1, p_2) is any point in a coordinate plane and O is the origin, then the vector $\vec{OP}$ is called the **position vector** of P, i.e. $\vec{OP} = p_1\vec{i} + p_2\vec{j}$, where p_1 and p_2 are the components of position vector.



If A (x_1, y_1) and B (x_2, y_2) are two points, then the vector $\vec{AB}$ can be determined by the position vectors of A and B.

From vector addition, $\vec{OA} + \vec{AB} = \vec{OB} \Rightarrow \vec{AB} = \vec{OB} - \vec{OA}$

Since $\vec{OA}$ is a position vector of A and $\vec{OB}$ is a position vector of B, then

$\vec{AB} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}$ or $\vec{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle$

Example 1:

Find the magnitude and unit vector in direction of vector $\vec{AB}$ from point A(5, -3) to B(9,7). Find the angle from $\vec{AB}$ to the positive x-axis.

Solution:

$\vec{AB} = \langle 9-5, 7-(-3) \rangle = \langle 4,10 \rangle$

Magnitude(length) = $\left|\vec{AB}\right| = \sqrt{4^2 + 10^2} = \sqrt{116} = 10.77$

Unit vector in direction of vector $\vec{AB}$ = $\dfrac{\vec{AB}}{\left|\vec{AB}\right|} = \dfrac{\langle 4,10 \rangle}{10.77} = \dfrac{4}{10.77}\vec{i} + \dfrac{10}{10.77}\vec{j}$

$\tan\theta = \dfrac{10}{4}$,

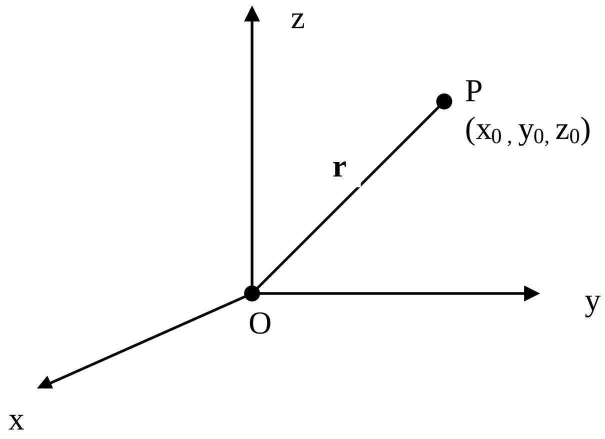
$\theta = \tan^{-1}(2.5) = 68.2°$

Exercises:

1. Give the vector for each of the following.
   (a) The vector from $A(2,-7)$ to $B(1,-3)$.
   (b) The vector from $P(1,-3)$ to $Q(2,-7)$.
   (c) The position vector for $P(-9,4)$.

2. Find the magnitude and unit vector in direction of vector $\vec{AB}$ from point A(2,3) to B(4,5). Find the angle from $\vec{AB}$ to the positive x-axis.

VECTORS IN THREE DIMENSIONS



If $P(x_0, y_0, z_0)$ is a point in the Cartesian coordinates, then the vector $\vec{r} = \vec{OP}$ from the origin to a point P is called the position vector of P.
We write the vector from the origin O to the point $P(x_0, y_0, z_0)$ as

$$\vec{r} = \vec{OP} = (x_0, y_0, z_0)$$

VECTOR OPERATIONS:

i. Vector Addition and Multiplication by Scalar

If $\vec{A} = (x_1\vec{i} + y_1\vec{j} + z_1\vec{k})$, and $\vec{B} = (x_2\vec{i} + y_2\vec{j} + z_2\vec{k})$

a. $\vec{A} \pm \vec{B} = (x_1 + x_2)\vec{i} + (y_1 + y_2)\vec{j} + (z_1 + z_2)\vec{k})$

b. $c\vec{A} = cx_1\vec{i} + cy_1\vec{j} + cz_1\vec{k})$

ii. Length (magnitude) of a Vector

If $\vec{A} = (x_1\vec{i} + y_1\vec{j} + z_1\vec{k})$, the magnitude of $\vec{A}$, $\left|\vec{A}\right| = \sqrt{x^2 + y^2 + z^2}$

iii. Vector between Two Points

If A (x_1, y_1, z_1) and B $(x_2, y_2 z_2)$ are 2 points in space,
Then the vector from A to B is

$$\vec{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \text{ or } \vec{AB} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k})$$

iv. Distance between Two Points

$$\left|\vec{AB}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

v. Direction of a Vector

Direction of $\vec{A} = \dfrac{\vec{A}}{\left|\vec{A}\right|}$ or Direction of $\vec{AB} = \dfrac{\vec{AB}}{\left|\vec{AB}\right|}$

Example 1:

Compute $\mathbf{a}+\mathbf{b}$, $\mathbf{a}-3\mathbf{b}$ and $|4\mathbf{a}+2\mathbf{b}|$

1. $\mathbf{a}=\langle -1,0,2\rangle, \mathbf{b}=\langle 4,3,2\rangle$ 2. $\mathbf{a}=3\vec{i}-\vec{j}+4\vec{k}$, $\mathbf{b}=5\vec{i}+\vec{j}$

Solution:

1.
$$
\begin{aligned}
a+b &= \langle -1+4,\ 0+3,\ 2+2\rangle \\
&= \langle 3,\ 3,\ 4\rangle
\end{aligned}
$$

$$
\begin{aligned}
a-3b &= \langle -1,\ 0,\ 2\rangle - 3\langle 4,\ 3,\ 2\rangle \\
&= \langle -1,\ 0,\ 2\rangle - \langle 12,\ 9,\ 6\rangle \\
&= \langle -1-12,\ 0-9,\ 2-6\rangle \\
&= \langle -13,\ -9,\ -4\rangle
\end{aligned}
$$

$$
\begin{aligned}
4a+2b &= \langle 4(-1,\ 0,\ 2)\rangle + 2\langle 4,\ 3,\ 2\rangle \\
&= \langle -4,\ 0,\ 8\rangle + \langle 8,\ 6,\ 4\rangle \\
&= \langle -4+8,\ 0+6,\ 8+4\rangle \\
&= \langle 4,\ 6,\ 12\rangle
\end{aligned}
$$

$$
\begin{aligned}
|4a+2b| &= \sqrt{(4)^2+(6)^2+(12)^2} \\
&= \sqrt{16+36+144} \\
&= \sqrt{196} \\
&= 14
\end{aligned}
$$

2.
$$
\begin{aligned}
a+b &= (3+5)\vec{i}+(-1+1)\vec{j}+(4+0)\vec{k} \\
&= 8\vec{i}+4\vec{k}
\end{aligned}
$$

$$
\begin{aligned}
a-3b &= (3\vec{i}+-\vec{j}+4\vec{k})-3(5\vec{i}+\vec{j}) \\
&= (3-15)\vec{i}+(-1-3)\vec{j}+(4-0)\vec{k} \\
&= 12\vec{i}-4\vec{j}+4\vec{k}
\end{aligned}
$$

$$
\begin{aligned}
4a+2b &= 4(3\vec{i}-\vec{j}+4\vec{k})+2(5\vec{i}+\vec{j}) \\
&= (12+10)\vec{i}+(-4+2)\vec{j}+(16+0)\vec{k} \\
&= 22\vec{i}-2\vec{j}+16\vec{k}
\end{aligned}
$$

$$
\begin{aligned}
|4a+2b| &= \sqrt{(22)^2+(-2)^2+(16)^2} \\
&= \sqrt{484+4+256} \\
&= \sqrt{744} \\
&= 27.28
\end{aligned}
$$

Example 2:

Given $\vec{A} = \vec{i} + \vec{j} + \vec{k}$, $\vec{B} = 2\vec{i} + 3\vec{k}$ and $\vec{C} = 3\vec{i} - \vec{j} + 2\vec{k}$ are the position vectors of point A, B, and C. Find a unit vector in the direction of $\vec{D}$ (if $\vec{D} = \vec{AB} - \vec{AC}$).

Solution:

$$\vec{A} = \vec{i} + \vec{j} + \vec{k} \qquad \Rightarrow A\,(1,1,1)$$

$$\vec{B} = 2\vec{i} + 3\vec{k} \qquad \Rightarrow B\,(2,0,3)$$

$$\vec{C} = 3\vec{i} - \vec{j} + 2\vec{k} \qquad \Rightarrow C\,(3,-1,2)$$

Vector, $\vec{D} = \vec{AB} - \vec{AC} = (\vec{i} - \vec{j} + 2\vec{k}) - (2\vec{i} - 2\vec{j} + \vec{k}) = -\vec{i} + \vec{j} + \vec{k}$

a unit vector, $\hat{D} = \dfrac{\vec{D}}{|\vec{D}|} = \dfrac{-\vec{i} + \vec{j} + \vec{k}}{\sqrt{1^2 + 1^2 + 1^2}} = \dfrac{-\vec{i}}{\sqrt{3}} + \dfrac{\vec{j}}{\sqrt{3}} + \dfrac{\vec{k}}{\sqrt{3}}$

Exercises:

1. Given the vectors **u** = 3**i** - **j** + 2**k** and **v** = 2**i** + 3**j** - **k**, find Cartesian forms of the vectors **u** + **v**, **u** - **v**, 2**u** + 3**v**.

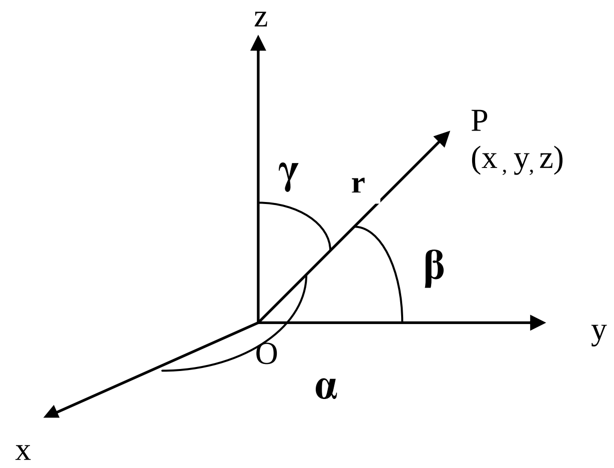
2. Suppose that **a** = 2**i** + **j**, **b** = **i** - 4**j** + **k** and **c** = **j** + **k**. What is the magnitude of 2**a** - **b** + **c**?

3. Find non-zero scalars α and β such that for all vectors $\vec{a}$ and $\vec{b}$ we have $\alpha\left(\vec{a} + 2b\right) - \beta\vec{a} + 4\vec{b} - \vec{a} = \vec{0}$

DIRECTION ANGLES AND DIRECTION COSINES:

The **direction angles** of a nonzero vector $\overrightarrow{OP}$ are the angles α, β and γ (in the interval [0, π]) that makes $\overrightarrow{OP}$ with the positive x-, y-, and z-axes.

The cosines of these direction angles, cos α, cos β, and cos γ are called the **direction cosines** of the vector $\overrightarrow{OP}$.



$\left|\overrightarrow{OP}\right| = \sqrt{x^2+y^2+z^2} = r$ or $r^2 = x^2+y^2+z^2$. We obtain

$$\boxed{\cos\alpha = \frac{x}{r}\ ,\ \cos\beta = \frac{y}{r}\ ,\ \cos\gamma = \frac{z}{r}}$$ (**direction cosines**)

By squaring the above expression and adding, we see that

$$\cos^2\alpha = \left(\frac{x}{r}\right)^2,\ \cos^2\beta = \left(\frac{y}{r}\right)^2,\ \cos^2\gamma = \left(\frac{z}{r}\right)^2,$$

$$\begin{aligned}\cos^2\alpha + \cos^2\beta + \cos^2\gamma &= \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 + \left(\frac{z}{r}\right)^2\\ &= \frac{\mathbf{x^2+y^2+z^2}}{\mathbf{r^2}}\\ &= \frac{\mathbf{r^2}}{\mathbf{r^2}}\\ &= \mathbf{1}\end{aligned}$$ (**identity**)

$\overrightarrow{OP} = \langle \mathbf{x, y, z}\rangle$

$= \langle \mathbf{r\cos}\alpha, \mathbf{r\cos}\beta, \mathbf{r\cos}\gamma\rangle$

Where, $x = \mathbf{r\cos}\alpha$, $y = \mathbf{r\cos}\beta$ and $z = \mathbf{r\cos}\gamma$

Thus,

$$\boxed{\alpha = \cos^{-1}\frac{x}{r}\ ,\ \beta = \cos^{-1}\frac{y}{r}\ ,\ \gamma = \cos^{-1}\frac{z}{r}}$$ (**direction angles ,** 0 ≤ α , β, γ ≤ π)

Example 1:
Find the direction cosines and direction angles of the vector .

a. $\vec{T} = \langle 3, 4, 5 \rangle$ b. $\vec{W} = 2\vec{i} + 3\vec{j} - 6\vec{k}$

Solution:

a. since $\left|\vec{T}\right| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50}$

$\cos\alpha = \dfrac{x}{r} = \dfrac{3}{\sqrt{50}}$ $\cos\beta = \dfrac{y}{r} = \dfrac{4}{\sqrt{50}}$ $\cos\gamma = \dfrac{z}{r} = \dfrac{5}{\sqrt{50}}$

and so

$\alpha = \cos^{-1}\left(\dfrac{3}{\sqrt{50}}\right) \approx 64.9^\circ$, $\beta = \cos^{-1}\left(\dfrac{4}{\sqrt{50}}\right) \approx 55.55^\circ$, $\gamma = \cos^{-1}\left(\dfrac{5}{\sqrt{50}}\right) \approx 45^\circ$

b. since $\left|\vec{W}\right| = \sqrt{2^2 + 3^2 + (-6)^2} = \sqrt{49} = 7$

$\cos\alpha = \dfrac{x}{r} = \dfrac{2}{7}$ $\cos\beta = \dfrac{y}{r} = \dfrac{3}{7}$ $\cos\gamma = \dfrac{z}{r} = \dfrac{-6}{7}$

and so

$\alpha = \cos^{-1}\left(\dfrac{2}{7}\right) \approx 73.4^\circ$, $\beta = \cos^{-1}\left(\dfrac{3}{7}\right) \approx 64.62^\circ$, $\gamma = \cos^{-1}\left(\dfrac{-6}{7}\right) \approx 149^\circ$

Example 2:
Let $\vec{u} = \vec{j} - 2\vec{k}$ and $\vec{v} = 2\vec{i} - \vec{j} + 3\vec{k}$. Find the direction angles of $\vec{u} - \vec{v}$.

Solution:

$\vec{u} - \vec{v} = \langle 0-2, 1-(-1), (-2)-3 \rangle = \langle -2, 2, -5 \rangle$

$\left|\vec{u} - \vec{v}\right| = \sqrt{(-2)^2 + 2^2 + (-5)^2} = \sqrt{33}$

$\cos\alpha = \dfrac{-2}{\sqrt{33}}$ $\cos\beta = \dfrac{2}{\sqrt{33}}$ $\cos\gamma = \dfrac{-5}{\sqrt{33}}$

and so

$\alpha = \cos^{-1}\left(\dfrac{-2}{\sqrt{33}}\right) \approx 110.37^\circ$, $\beta = \cos^{-1}\left(\dfrac{2}{\sqrt{33}}\right) \approx 69.63^\circ$, $\gamma = \cos\left(\dfrac{-5}{\sqrt{33}}\right) \approx 150.50^\circ$

Example 3:

Given vectors $\vec{u} = 2\vec{i} + 3\vec{j} - 4\vec{k}$, $\vec{v} = 4\vec{i} - 2\vec{j} + \vec{k}$ and $\vec{w} = \vec{i} + 3\vec{j} - 4\vec{k}$, find the direction angles for $\vec{u} + \vec{v} - \vec{w}$.

Solution:

Exercises:

1. Find the direction cosines and direction angles of the vector.

   a. $\vec{T} = \langle 2, 4, 3 \rangle$ b. $\vec{W} = \vec{i} - 3\vec{j} + 2\vec{k}$

2. Let $\vec{u} = 2\vec{j} + \vec{k}$ and $\vec{v} = -2\vec{i} + \vec{j} + 4\vec{k}$. Find the direction angles of $\vec{u} - \vec{v}$.

PRODUCTS OF TWO VECTORS:

Scalar Product (Dot or Inner Product) of 2 Vectors

Definition:

If $\vec{A} = \langle a_1, a_2, a_3 \rangle$ and $\vec{B} = \langle b_1, b_2, b_3 \rangle$, then the dot product of $\vec{A}$ and $\vec{B}$ is the number $\vec{A} \bullet \vec{B}$ given by

$\vec{A} \bullet \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$ (**in components**)

$\vec{A} \bullet \vec{B} = |a||b| \cos\theta$ (**geometric interpretation**)

where **θ** ($0 \leq \theta \leq \pi$) is the angle between the two vectors.

Points to Ponder :

i. The scalar product of two vectors gives a **number** (a scalar).

ii. Scalar product is only defined on two vectors.

Properties of Scalar Product

1. $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$
2. $\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$
3. $m(\vec{A} \bullet \vec{B}) = (m\vec{A}) \bullet \vec{B} = \vec{A} \bullet (m\vec{B}) = (\vec{A} \bullet \vec{B})m$, where m is any scalar
4. $\vec{i} \bullet \vec{i} = \vec{k} \bullet \vec{k} = \vec{j} \bullet \vec{j} = 1$, $\vec{i} \bullet \vec{j} = \vec{j} \bullet \vec{k} = \vec{k} \bullet \vec{i} = 0$
5. If $\vec{A} = A_1 \vec{i} + A_2 \vec{j} + A_3 \vec{k}$ and $\vec{B} = B_1 \vec{i} + B_2 \vec{j} + B_3 \vec{k}$

   then $\vec{A} \bullet \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$

   $\vec{A} \bullet \vec{A} = A_1^2 + A_2^2 + A_3^2$

   $\vec{B} \bullet \vec{B} = B_1^2 + B_2^2 + B_3^2$
6. If θ is the angle between two vectors $\vec{A}$ and $\vec{B}$, then $\cos\theta = \dfrac{\vec{A} \bullet \vec{B}}{|\vec{A}||\vec{B}|}$.
7. $\vec{A}$ and $\vec{B}$ are orthogonal (perpendicular) iff $\vec{A} \bullet \vec{B} = 0$.

Example 1:

Find $\vec{A} \bullet \vec{B}$.

1. $\vec{A} = \langle 5, 0, -2 \rangle$, $\vec{B} = \langle 3, -1, 10 \rangle$

. $\vec{A} = 4\vec{j} - 3\vec{k}$, $B = 2\vec{i} + 4\vec{j} + 6\vec{k}$

3. $|\vec{A}| = 12$, $|\vec{B}| = 15$, the angle between $\vec{A}$ and $\vec{B}$ is $\dfrac{\pi}{6}$

Solution:

1. $\vec{A} \bullet \vec{B} = 5(3) + 0(-1) + (-2)(10)$
$= 15 --20 = -5$

2. $\vec{A} \bullet \vec{B} = 0(2) + 4(4) + (-3)(6)$
$= 16 - 18 = -2$

3. $\vec{A} \bullet \vec{B} = 12 \times 15 \cos\left(\frac{\pi}{6}\right)$
$= 155.88$

Example 2:

Given $\vec{A} = 7\vec{i} + 4\vec{j} - 2\vec{k}$ and $\vec{B} = -\vec{i} + 4\vec{j} - 5\vec{k}$, find the angle between vector $\vec{A}$ and vector $\vec{B}$.

Solution:

Exercises:

1. What is a·b if a = 3i-j and b = 2i + j + 4k?

2. Given vectors **u** and **v** such that |**u**| = 1, |**v**| = 3 and the angle between **u** and **v** is 45°. Calculate **u** · **v**.

3. Given that **u** is a vector of magnitude 2, **v** is a vector of magnitude 3 and the angle between them when placed tail to tail is 45°, what is **u** · **v** (to one decimal place)?

4. Show that vectors v = <3 , - 4> and u = <4 , 3> are orthogonal.

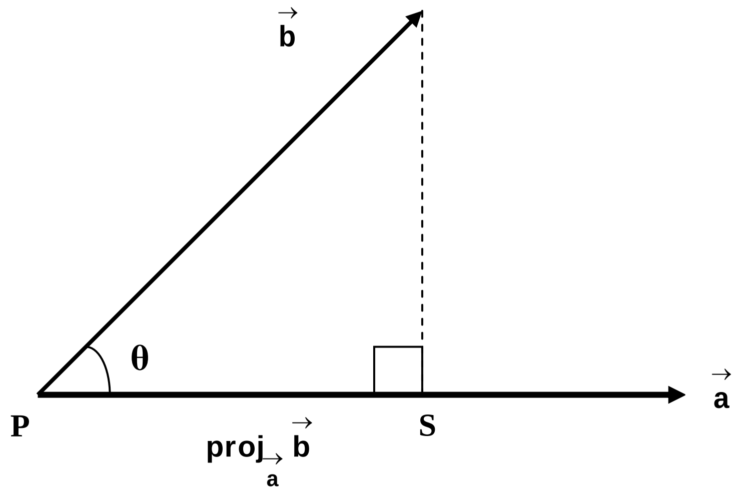
5. Determine the angle between $\vec{a} = \langle 3, -4, -1 \rangle$ and $\vec{b} = \langle 0, 5, 2 \rangle$.

6. Given vectors; **v** = <10 , - 5> and **u** = <2 , a>. Find value of a so that vectors **v** and **u** are orthogonal.

PROJECTIONS (APPLICATIONS OF THE DOT PRODUCTS):

One important use of dot products is in projections. Let say, given two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, we want to determine the projection of $\vec{\mathbf{b}}$ onto $\vec{\mathbf{a}}$. So, to get the projection of $\vec{\mathbf{b}}$ onto $\vec{\mathbf{a}}$, we drop a perpendicular line segment from the end of $\vec{\mathbf{b}}$ to the line that is parallel to $\vec{\mathbf{a}}$.

The projection of $\vec{\mathbf{b}}$ onto $\vec{\mathbf{a}}$ is the ***length*** of the segment PS .



$$\frac{\left|\vec{\mathbf{PS}}\right|}{\left|\vec{\mathbf{b}}\right|}=\cos\theta \;\Rightarrow\; \left|\vec{\mathbf{PS}}\right|=\left|\vec{\mathbf{b}}\right|\cos\theta=\left|\vec{\mathbf{b}}\right|\cos\theta\frac{\left|\vec{\mathbf{a}}\right|}{\left|\vec{\mathbf{a}}\right|}=\frac{\left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right|\cos\theta}{\left|\vec{\mathbf{a}}\right|}=\frac{\vec{\mathbf{a}}\bullet\vec{\mathbf{b}}}{\left|\vec{\mathbf{a}}\right|}$$

$\therefore$ The scalar projection of vector $\vec{b}$ onto $\vec{a}=\dfrac{\vec{a}\bullet\vec{b}}{\left|\vec{a}\right|}$

The scalar projection of $\vec{\mathbf{b}}$ onto $\vec{\mathbf{a}}$ is given by $\operatorname{comp}_{\vec{a}}\vec{b}=\dfrac{\vec{a}\bullet\vec{b}}{\left|\vec{a}\right|}$. This quantity is also called the component of $\vec{\mathbf{b}}$ in the $\vec{\mathbf{a}}$ direction (hence the notation is comp). And, the vector projection is merely the unit vector **a**/|**a**| times the scalar projection of $\vec{\mathbf{b}}$ onto $\vec{\mathbf{a}}$:

$$\operatorname{proj}_{\vec{a}}\vec{b}=\frac{\vec{a}\bullet\vec{b}}{\left|\vec{a}\right|}\frac{\vec{a}}{\left|\vec{a}\right|}=\frac{\vec{a}\bullet\vec{b}}{\left|\vec{a}\right|^2}\vec{a}$$

Note:

Scalar Projection

$$\text{comp}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\bullet\vec{b}}{|\vec{a}|}$$

Vector Projection

$$\text{proj}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\bullet\vec{b}}{|\vec{a}|}\frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{\vec{a}\bullet\vec{b}}{|\vec{a}|^2}\vec{a}$$

Example:

Given $\vec{u} = 2\vec{i} - \vec{j} + \vec{k}$, $\vec{v} = \vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{w} = 2\vec{i} + 2\vec{j} + 2\vec{k}$.
Find,

i. acute angle between $2\vec{w}$ and $\vec{u} + \vec{v}$

ii. the scalar projection of $2\vec{w}$ onto $\vec{u} + \vec{v}$

iii. the vector projection of $2\vec{w}$ onto $\vec{u} + \vec{v}$

Solution:

i. $2\vec{w} = 2\langle 2, 2, 2\rangle = \langle 4, 4, 4\rangle$

$\vec{u} + \vec{v} = \langle 2, -1, 1\rangle + \langle 1, 2, -3\rangle = \langle 3, 1, -2\rangle$

$$\cos\theta = \frac{\langle 4, 4, 4\rangle \bullet \langle 3, -1, 2\rangle}{\left|\sqrt{4^2 + 4^2 + 4^2}\right|\left|\sqrt{3^2 + (-1)^2 + 2^2}\right|} = \frac{12 - 4 + 8}{\sqrt{48}\sqrt{14}}$$

$$\cos\theta = \frac{8}{\sqrt{48}\sqrt{14}} = 0.30861$$

$$\theta = \cos^{-1} 0.30861 = 72.024^\circ$$

ii. $2\vec{w} = 2\langle 2, 2, 2\rangle = \langle 4, 4, 4\rangle$

$\vec{u} + \vec{v} = \langle 2, -1, 1\rangle + \langle 1, 2, -3\rangle = \langle 3, 1, -2\rangle$

scalar projection of $2\vec{w}$ onto $\vec{u} + \vec{v}$

or

$$\text{comp}_{\vec{u}+\vec{v}} 2\vec{w} = \frac{\langle 4,4,4\rangle \bullet \langle 3,-1,2\rangle}{\left\|\sqrt{3^2+(-1)^2+2^2}\right\|} = \frac{12-4+8}{\sqrt{14}} = \frac{8}{\sqrt{14}}$$

iii. vector projection of $2\vec{w}$ onto $\vec{u}+\vec{v}$

$$\text{proj}_{\vec{u}+\vec{v}} 2\vec{w} = \frac{8}{\sqrt{14}} \frac{\langle 3,-1,2\rangle}{\sqrt{14}}$$
$$= \frac{8}{14}\langle 3,-1,2\rangle$$
$$= \frac{4}{7}\langle 3,-1,2\rangle$$

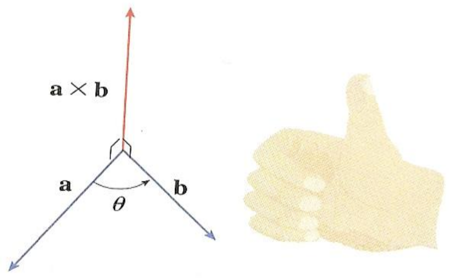
Exercises:

1. Determine the vector projection of $\vec{b} = \langle 2,1,-1\rangle$ onto $\vec{a} = \langle 1,0,-2\rangle$.

2. Determine the vector projection of $\vec{a} = \langle 1,0,-2\rangle$ onto $\vec{b} = \langle 2,1,-1\rangle$.

VECTOR PRODUCT (CROSS PRODUCT) OF 2 VECTORS:

The **cross product** **a** X **b** of two vectors **a** and **b**, unlike the dot product, is a **vector**. The **a** X **b** is defined only when **a** and **b** are three-dimensional vectors.



The **right-hand rule**: If you align the fingers of your right hand along the vector **a** and bend your fingers around in the direction of rotation from **a** toward **b** (through an angle less than 180°), your thumb will point in the direction of **a** x **b**.

Definition: If $\mathbf{a} = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \rangle$ and $\mathbf{b} = \langle \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \rangle$, then the cross product of **a** and **b** is the vector

$$\mathbf{a} \times \mathbf{b} = \langle \mathbf{a}_2\mathbf{b}_3 - \mathbf{a}_3\mathbf{b}_2, \mathbf{a}_3\mathbf{b}_1 - \mathbf{a}_1\mathbf{b}_3, \mathbf{a}_1\mathbf{b}_2 - \mathbf{a}_2\mathbf{b}_1 \rangle$$

or we can use the notation of determinants to make this definition easier to remember:

$$a \times b = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Properties of Cross Product

For any vectors a, b and c in V_3 and any scalar m, the following hold:

i. $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$ (anticommutative)

ii. $(d\mathbf{a}) \times \mathbf{b} = d(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (d\mathbf{b})$ where d is a constant

iii. $\vec{i} \times \vec{j} = -\vec{j} \times \vec{i} = \vec{k}$

$\vec{j} \times \vec{k} = -\vec{k} \times \vec{j} = \vec{i}$

$\vec{k} \times \vec{i} = -\vec{i} \times \vec{k} = \vec{j}$

$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$

iv. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (distributive law)

$(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}$ (distributive law)

v. $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}$ (scalar triple product)

vi. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c}$ (vector triple product)

vii Two nonzero vectors **a** , **b** $\in V_3$, if θ is the angle between **a** and **b** ($0 \le \theta \le \Pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

viii. Two nonzero vectors **a** , **b** $\in V_3$ are ***parallel*** if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$

Example 2:

If $\vec{a} = \langle 1, 3, 4 \rangle$ and $\vec{b} = \langle 2, 7, -5 \rangle$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \vec{k}$$

$$= (-15 - 28)\,\vec{i} - (-5 - 8)\,\vec{j} + (7 - 6)\,\vec{k} = 43\,\vec{i} + 13\,\vec{j} + \vec{k}$$

Example 3:

Use the cross product to determine the angle between the vectors

$$\vec{a} = \langle 2, -1, 0 \rangle,\ \vec{b} = \langle 1, 0, 3 \rangle$$

Solution:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 0 \\ 1 & 0 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} \vec{k}$$

$$= (-3)\,\vec{i} - (6)\,\vec{j} + (1)\,\vec{k} = -3\,\vec{i} - 6\,\vec{j} + \vec{k}$$

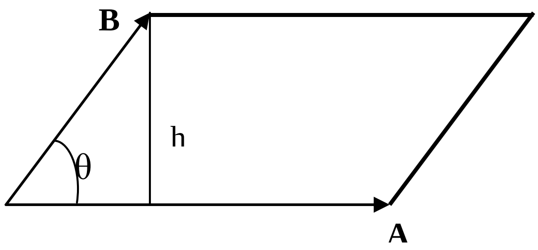
$$\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right| \sin\theta$$

$$\sin\theta = \frac{\left|\vec{a} \times \vec{b}\right|}{\left|\vec{a}\right|\left|\vec{b}\right|} = \frac{\sqrt{(-3)^2 + (-6)^2 + (1)^2}}{\sqrt{(2)^2 + (-1)^2}\sqrt{(1)^2 + (3)^2}} = \frac{\sqrt{9 + 36 + 1}}{\sqrt{5}\sqrt{10}} = \frac{\sqrt{46}}{\sqrt{5}\sqrt{10}}$$

$$\theta = \sin^{-1} \frac{\sqrt{46}}{\sqrt{5}\sqrt{10}} = 73.57^\circ$$

SIMPLE APPLICATION OF DOT, CROSS AND TRIPLE PRODUCT:

i. <u>Parallelogram</u>

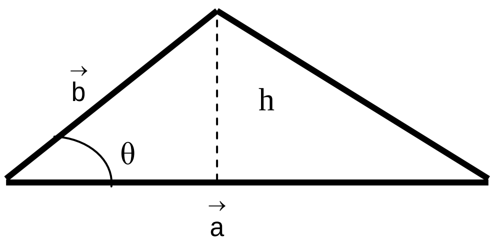


Area of a Parallelogram = $|\mathbf{A}|\,h$

$= |\mathbf{A}|\,|\mathbf{B}|\sin\theta$

$= |\mathbf{A} \times \mathbf{B}|$

ii. <u>Triangle</u>



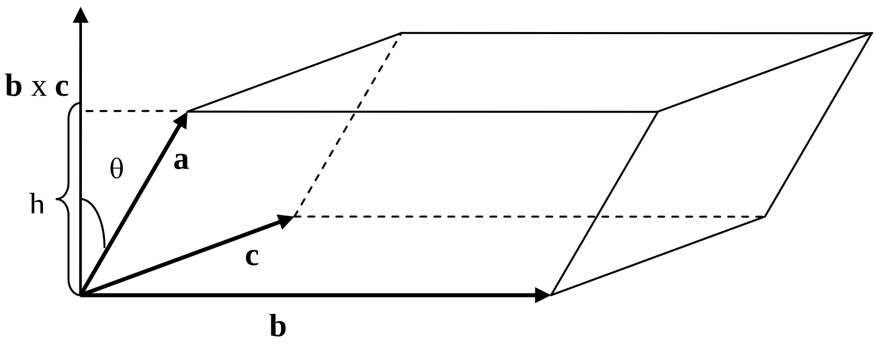
Area of a Triangle = ½ x base x height

$= \frac{1}{2} \times |\vec{a}|\,h$

$= \frac{1}{2} \times |\vec{a}|\,|\vec{b}|\sin\theta$

$= \frac{1}{2}|\vec{a} \times \vec{b}|$

iii. <u>Parallelepiped</u>



The volume of a parallelepiped, V determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

A (area of the base parallelogram) = $|\mathbf{b} \times \mathbf{c}|$

θ (angle between **a** and **b** x **c**)

h(height of the parallelepiped) = $|\mathbf{a}|\,|\cos\theta|$

$|\mathbf{a}||\mathbf{b}|\cos\theta = \mathbf{a} \bullet \mathbf{b}$

Volume of the parallelepiped,V = Ah

$= |\mathbf{b} \times \mathbf{c}|\,|\mathbf{a}|\cos\theta|$

$= |(\mathbf{b} \times \mathbf{c}) \bullet \mathbf{a}|$

or, $V = |\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})|$

iv. Condition

a. parallelism $\Rightarrow$ if $\vec{a}$ and $\vec{b}$ are parallel, then $\vec{a} \times \vec{b} = 0$

b. perpendicular(orthogonal) $\Rightarrow$ if $\vec{a}$ and $\vec{b}$ are perpendicular , then $\vec{a} \bullet \vec{b} = 0$

c. coplanar $\Rightarrow$ if the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ lie on the same plane, then $\vec{a} \bullet (\vec{b} \times \vec{c}) = 0$

Example 4:

Find the area of the parallelogram with two adjacent sides formed by the vectors $\vec{a} = \langle 1, 2, 3 \rangle$ and $\vec{b} = \langle 4, 5, 6 \rangle$.

Solution:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \vec{k}$$

$$= (12 - 15)\,\vec{i} - (6 - 12)\,\vec{j} + (5 - 8)\,\vec{k} = -3\,\vec{i} + 6\,\vec{j} - 3\,\vec{k} = \langle -3, 6, -3 \rangle$$

$$\left| \vec{a} \times \vec{b} \right| = \sqrt{(-3)^2 + 6^2 + (-3)^2}$$

$$= 7.35 \text{ unit}^2$$

Example 5:

Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$ and $R(1, -1, 1)$.

Solution: The vector $\vec{PQ} \times \vec{PR}$ is perpendicular to both $\vec{PQ}$ and $\vec{PR}$ and is therefore perpendicular to the plane through P, Q and R.

$$\vec{PQ} = (-2 - 1)\,\vec{i} + (5 - 4)\,\vec{j} + (-1 - 6)\,\vec{k} = -3\,\vec{i} + \vec{j} - 7\,\vec{k}$$

$$\vec{PR} = (1 - 1)\,\vec{i} + (-1 - 4)\,\vec{j} + (1 - 6)\,\vec{k} = -5\,\vec{j} - 5\,\vec{k}$$

We compute the cross product of these vectors:

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= (-5 - 35)\,\vec{i} - (-15 - 0)\,\vec{j} + (15 - 0)\,\vec{k} = -40\,\vec{i} - 15\,\vec{j} + 15\,\vec{k}$$

So the vector $\langle -40, -15, 15 \rangle$ is perpendicular to the given plane.

Example 6:
Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$ and $R(1, -1, 1)$.

Solution:

In the previous example, we computed that $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -40, -15, 15 \rangle$. The area of the parallelogram with adjacent sides $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is the length of this cross product:

$$\frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{(-40)^2 + (-15)^2 + 15^2} = \frac{1}{2}\sqrt{2050} = \frac{5}{2}\sqrt{82}\ \text{unit}^2$$

Example 7:
Find the volume of a parallelepiped determined by the vectors $\vec{a} = \langle 3,4,-1 \rangle$, $\vec{b} = \langle 1,3,2 \rangle$, and $\vec{c} = \langle 2,2,1 \rangle$.

Solution:

Volume of parallelepiped, $V = |\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})|$

$$= \begin{vmatrix} 3 & 4 & -1 \\ 1 & 3 & 2 \\ 2 & 2 & 1 \end{vmatrix}$$

$$= 3(3-4) - 4(1-4) - 1(2-6)$$

$$= 3(-1) - 4(-3) - 1(-4)$$

$$= -3 + 12 + 4$$

$$= 13\ \text{unit}^3$$

Example 8:
Use the scalar triple product to show that the vectors $\vec{a} = \langle 1, 4, -7 \rangle, \vec{b} = \langle 2, -1, 4 \rangle$ and $\vec{c} = \langle 0, -9, 18 \rangle$ are coplanar.

Solution:

$$\vec{a} \bullet (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7\begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

= 1(-18+36) □ 4(36) □7(-18) = 0

Therefore, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

Exercises:

1. If $\vec{a} = \langle 2,1,-1 \rangle$ and $\vec{b} = \langle -3,4,1 \rangle$ compute each of the following.

    i. $\vec{a} \times \vec{b}$ ii. $\vec{b} \times \vec{a}$

2. Given **u** = **i** - 2**j** + 2**k** and **v** = 4**i** - 3**k**. Calculate

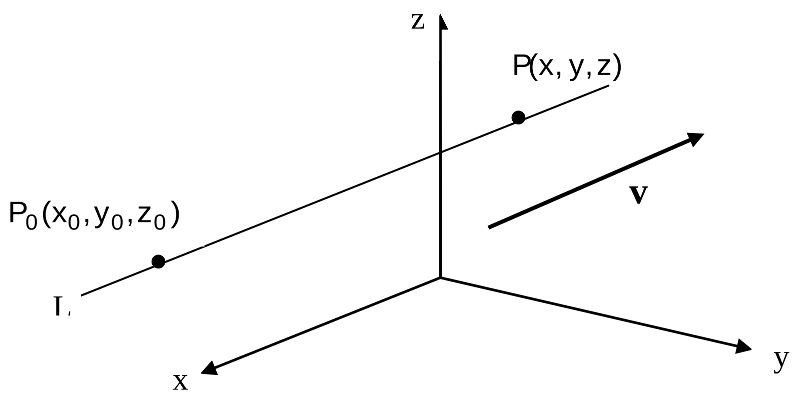
    i. **u** × **v** ii. the sine of the angle between them

3. Calculate the vector cross product **a**×**b** when **a**=3**i**+**j**-2**k** and **b**=4**i**-**j**

4. A plane is defined by any three points that are in the plane. If a plane contains the points $P = (1,0,0)$, $Q = (1,1,1)$ and $R = (2,-1,3)$ find a vector that is orthogonal to the plane.

SIMPLE GEOMETRIC APPLICATIONS OF SCALAR PRODUCT AND VECTOR PRODUCT:

Equation of Lines



Suppose L is a line in space. L passes through a point $P_0(x_0, y_0, z_0)$ and is parallel to a vector $\vec{\mathbf{v}} = A\vec{i} + B\vec{j} + C\vec{k}$. Let $P(x, y, z)$ be any point on the line L. Then $\overrightarrow{P_0P}$ is parallel to the vector $\vec{\mathbf{v}}$. Further, recall that two vectors are parallel if and if only one is a scalar multiple of the other. This says that

Vector equation: $\overrightarrow{P_0P} = t\,\vec{\mathbf{v}}$, for some scalar t.

$$\Rightarrow (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k} = t(A\vec{i} + B\vec{j} + C\vec{k})$$

Parametric equations: $(x - x_0) = tA, \quad (y - y_0) = tB, \quad (z - z_0) = tC$

or $\quad x = x_0 + tA, \quad y = y_0 + tB, \quad z = z_0 + tC$

Cartesian/Symmetric equations: $\dfrac{x - x_0}{A} = \dfrac{y - y_0}{B} = \dfrac{z - z_0}{C}$

Example 1:

Find a vector equation and parametric equations for the line that passes through the points (5,1,3) and is parallel to the vector $\vec{i} + 4\vec{j} - 2\vec{k}$.

Solution:

Let P_0 (5, 1, 3) , P(x, y, z) so $\overrightarrow{P_0P} = \langle x - 5, y - 1, z - 3\rangle$ and $\vec{\mathbf{v}} = \vec{i} + 4\vec{j} - 2\vec{k}$.

Vector equation: $\overrightarrow{P_0P} = t\,\vec{\mathbf{v}}$

$$\langle x - 5, y - 1, z - 3\rangle = t\,(\vec{i} + 4\vec{j} - 2\vec{k})$$

Parametric equations: $(x - 5) = t, \quad (y - 1) = 4t, \quad (z - 3) = -2t$

Example 2:

i. Find parametric equations and Cartesian equations of the line that passes through the point A(2, 4, □3) and B(3, □1,1).

ii. At what point does this line intersect the xy-plane?

Solution:

i. First, we need to find a vector that is parallel to the given line.

$\overrightarrow{AB} = \langle 3-2, -1-4, 1-(-3)\rangle = \langle 1, -5, 4\rangle$

Taking the point (2, 4, □3) as P_0 and (x, y, z) as P, we see that parametric equations are

x = 2 + t, y = 4 – 5t, z = –3+4t

and Cartesian equations are

$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{z+3}{4}$$

ii. The line intersects the xy-plane when z = 0, so we put z = 0 in the parametric equation for z = –3 + 4t and obtain

$$-3 + 4t = 0, \quad t = \frac{3}{4}.$$

Substitute $t = \frac{3}{4}$ into the parametric equations for x and y.

So, the line intersects the xy-plane at the point $\left(\frac{11}{4}, \frac{1}{4}, 0\right)$.

Note:

i. The lines L_1 and L_2 are parallel whenever $\vec{a}$ and $\vec{b}$ are parallel.

ii. If lines L_1 and L_2 intersect, then

a. the angle between L_1 and L_2 is θ.

b. the lines L_1 and L_2 are orthogonal(perpendicular) whenever $\vec{a}$ and $\vec{b}$ are orthogonal(perpendicular).

Exercises:

1. Write down the equation of the line that passes through the points $(2,-1,3)$ and $(1,4,-3)$. Write down all three forms of the equation of the line.

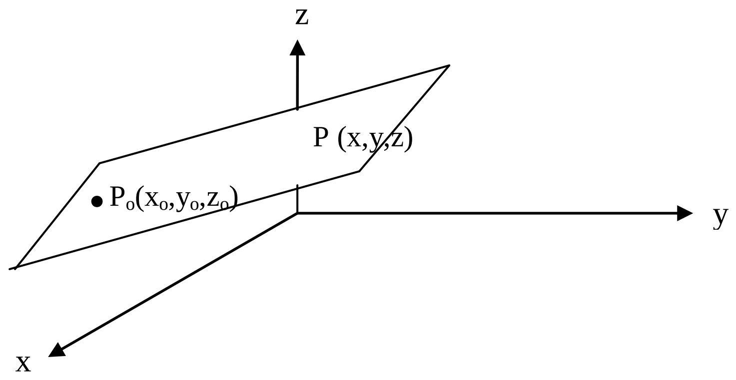
2. Find the equation of the line through (1, 2, 1) parallel to the line

$$\frac{x-2}{3} = \frac{y-1}{1} = \frac{z-5}{4}$$

3. Find the parametric and vector equation of the line through (1, 1, 1) that is perpendicular to the lines (1) and (2):

$$(1): \vec{r} = \vec{i} - \vec{j} + \vec{k} \qquad (2): \frac{x}{4} = y + 1 = \frac{z-2}{3}$$

Equation of Planes



Suppose $P_0 = (x_0, y_0, z_0)$ be a point on a plane and a vector $\vec{N} = A\vec{i} + B\vec{j} + C\vec{k}$ is normal (perpendicular/orthogonal) to the plane. Let P (x, y,z) be an arbitrary point in the plane. The vector $\overrightarrow{P_0P}$ is a vector on the plane and perpendicular to $\vec{N}$.

So,

$$\overrightarrow{P_0P} \bullet \vec{N} = 0$$

$$\Rightarrow \quad (x - x_0)A + (y - y_0)B + (z - z)C = 0$$

$$\Rightarrow \quad Ax + By + Cz = D \quad \text{where} D = Ax_0 + By_0 + Cz_0$$

Equation of the plane passes through $P_0 = (x_0, y_0, z_0)$ and normal to the vector $\vec{N} = A\vec{i} + B\vec{j} + C\vec{k}$: $\quad Ax + By + Cz = D$

Example 1:
Finding the Equation of a Plane Given Three Points

Find an equation of the plane that passes through the points P(1, 3, 2), Q(3,-1,6) and R(5,2,0).

Solution:

First, we'll need to find a vector normal to the plane. Notice that two vectors lying in the plane are $\quad \overrightarrow{PQ} = \langle 2, -4, 4 \rangle$ and $\overrightarrow{QR} = \langle 2, 3, -6 \rangle$

Since $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ lie in the plane, their cross product $\overrightarrow{PQ} \times \overrightarrow{QR}$ is orthogonal to the plane and can be taken as the normal vector.

$$\vec{N} = \overrightarrow{PQ} \times \overrightarrow{QR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 4 \\ 2 & 3 & -6 \end{vmatrix} = 12\vec{i} + 20\vec{j} + 14\vec{k}$$

With the point P (1, 3, 2) and A(x, y, z) and the normal vector n, an equation of the plane is

$$\overrightarrow{PA} \bullet \vec{N} = 0$$

$$\langle x - 1, y - 3, z - 2 \rangle \bullet \langle 12, 20, 14 \rangle = 0$$

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$$

$$12x + 20y + 14z = 100$$

The Equation of a Plane Given a Point and a Parallel Plane

Find an equation for the plane through the point (1, 4, –5) and parallel to the plane defined by 2x – 5y +7z = 12.

Solution:

First, notice that a normal vector to the given plane is $\langle 2, -5, 7\rangle$. Since the two planes are to be parallel, this vector is also normal to the new plane.

The equation of the plane,

Let P(1, 4, –5) , A(x, y, z) and $\vec{N} = \langle 2, -5, 7\rangle$

$$\overrightarrow{PA} \bullet \vec{N} = 0$$

$$\langle x-1, y-4, z+5\rangle \bullet \langle 2, -5, 7\rangle = 0$$

$$2(x-1) - 5(y-4) + 7(z+5) = 0$$

$$2x - 5y + 7z = -53$$

Example 3:

Given 3 points A (0, -2, -3) , B (3, 4, 6) and C(1, 2 , 3) , find

(i) the area of triangle ABC
(ii) the equation of plane ABC
(iii) show that A, B ,C are coplanar.

Exercises:

1. Find the Cartesian equation of the plane containing the three points *A*(1, 3,-1), *B*(1, 1, 5) and *C*(-2, 1, 0).

2. Find a equation of the plane that passes through *C*(2, 1, 5) and is perpendicular to the line through *A*(0, 1, 1) and *B*(1,-1,-1).

3. Let *A*(-1, 3, 1), *B*(2,-2, 0) and *C*(1, 1,-1) be three points in space.

   a. Give an equation for the plane through *A*, *B* and *C*.
   b. Write down an equation for the line perpendicular to the plane through *A*, *B* and *C* passing through *A*.

Exercise 2.1

Given $\vec{A} = 2\vec{i} - \vec{j} + \vec{k}$, $\vec{B} = 3\vec{i} + \vec{j} + 2\vec{k}$

(i) Find the projection of $\vec{A}$ onto $\vec{B}$

(ii) If $\vec{C}$ is a vector perpendicular to $\vec{A}$ and $\vec{B}$

(a) find the unit vector of $\vec{C}$.

(b) find the direction cosine of $\vec{C}$.

Answers:

i. projection of $\vec{A}$ on $\vec{B} = \dfrac{\vec{A} \bullet \vec{B}}{|\vec{B}|} = \dfrac{7}{\sqrt{14}}$

ii. (a) $\vec{C} = \vec{A} \times B = -3\vec{i} - \vec{j} + 5\vec{k}$

(b) $\cos\alpha = \dfrac{-3}{\sqrt{35}}$; $\cos\beta = -\dfrac{1}{\sqrt{35}}$; $\cos\gamma = \dfrac{5}{\sqrt{35}}$

Exercise 2.2

(a) Given P(2, -1, 1), Q(4,-2, 2) and R(3, 1, -2). Find,

(i) the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

(ii) the equation of plane passes through point A and parallel to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

(iii) Given S(λ, -5, 6), find a such that $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$ are coplanar.

(b) Find the area of triangle whose vertices are P(3, -1, -1), Q(1, 1, 1) and R(4, -1, 1).

Answers:

(a) i. $\theta = \cos -1 (-0.327) = 109.1\,°$

ii. Equation of plane : $x + 7y + 5z = 0$

iii. $\lambda = 5$

(b) area of triangle = $\dfrac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$

$= \dfrac{1}{2}\sqrt{56}$

Exercise 2.3

(a) Given P(-2, 3, 1), Q(1, 5, 1) and R(4, -3, 0). Find the area of parallelogram spanned by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

(b) Given $\vec{u} = \vec{i} - \vec{j} + \vec{k}$, $\vec{v} = 2\vec{i} + \vec{j} - 2\vec{k}$, and $\vec{w} = 5\vec{i} - 2\vec{j} + \vec{k}$. Find,

(i) $\vec{u} \bullet \vec{v}$

(ii) $\vec{u} \bullet (\vec{v} \times \vec{w})$

(iii) $\vec{u} \times (\vec{v} \times \vec{w})$

Answers:

(a) area $=|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{913}$

(b) (i) -1

(ii) 0

(iii) $21\vec{i} + 6\vec{j} - 15\vec{k}$

Exercise 2.4

Given A(1, -1, -1), B(2, 5, 1) and C(3, 2, -3). Find the equation of the line through A and perpendicular to the plane containing point A, B and C.

Answers:

equation of line : parametric ➔ x -1 = -18t
y + 1 = 8t
+ 1 = -9t

Exercise 2.5

The position vectors of points A, B, C are:

$$\overrightarrow{OA} = -2\vec{i} + 3\vec{j} - 7\vec{k}$$
$$\overrightarrow{OB} = \vec{i} + 7\vec{j} + 5\vec{k}$$
$$\overrightarrow{OC} = 4\vec{i} + 3\vec{j} + \vec{k}$$

(i) Calculate $\left|\overrightarrow{AB}\right|$, $\left|\overrightarrow{AC}\right|$ and the scalar product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ hence deduce that

$$\angle BAC = \cos^{-1}\left(\frac{57}{65}\right)$$

(ii) Find the direction cosines and direction angles for a vector from point A to point B.

Answers:

(i) $\left|\overrightarrow{AB}\right| = 13$; $\left|\overrightarrow{AC}\right| = 10$; $\overrightarrow{AB} \bullet \overrightarrow{AC} = 114 = \left|AB\right| \bullet \left|AC\right| \cos BAC$

(ii) Direction cosines : $\cos\alpha = \frac{3}{13}$; $\cos\beta = \frac{4}{13}$; $\cos\gamma = \frac{12}{13}$

Exercise 2.6

Given A(-2, 0, -3), B(1,-2, 1), C(-2, -13/5, 26/5), D(16/5, -13/5 , 0). Find the equation of the plane passing through points A and B that is parallel to $\overrightarrow{CD}$.

Answers:

Equation of plane passing through points A and B and // to $\overrightarrow{CD}$:

$$\left(2\vec{i} + 7\vec{j} + 2\vec{k}\right) \bullet \vec{n} = -10$$

Exercise 2.7

Find the equation of the line that passes through points A(1, 2, 0) and B(2, 1, -1) in the vector, parametric and Cartesian form.

Answers:

i. vector eq. of a line : $\vec{r} = (1-\lambda)\vec{i} + (2+\lambda)\vec{j} + \lambda\vec{k}$

ii. para.eq : $x = 1-\lambda$
$y = 2 + \lambda$
$z = \lambda$

iii. Cartesian : $\lambda = (1-x) = (y-2) = z$

Exercise 2.8

The position vectors $\vec{u}$ and $\vec{v}$ of points A and B are $\vec{i} - \vec{k}$ and $3\vec{i} + \lambda\vec{j} - \vec{k}$ respectively, where λ is a scalar

(i) Find $\vec{u} \times \vec{v}$ in terms of λ

(ii) Given that the area of ΔOAB is $\sqrt{3}$, find the possible values of λ.

Answers:

(i) $\vec{u} \times \vec{v} = \lambda\vec{i} - 2\vec{j} + \lambda\vec{k}$

(ii) area OAB = $\frac{1}{2}\sqrt{4+2\lambda^2} = \sqrt{3}$

then, $\lambda = \pm 2$.

Exercise 2.9

Given the vectors

$\vec{u} = 6\vec{i} - 4\vec{j} + \vec{k}$

$\vec{v} = 8\vec{i} + 5\vec{j} - 3\vec{k}$

$\vec{w} = 2\vec{i} + 8\vec{j} - 5\vec{k}$

Evaluate

(i) $\vec{u} \times (\vec{v} \times \vec{w})$

(ii) $\vec{u} \bullet (\vec{v} \times \vec{w})$

Hence, determine the volume of the parallelepiped with the edges represented by $\vec{u}$, $\vec{v}$ and $\vec{w}$ as above.

Answers:

(i) $\vec{u} \times (\vec{v} \times \vec{w}) = -250I - 325j + 200k$

(ii) $\vec{u} \bullet (\vec{v} \times \vec{w}) = -88$ ➔ volume =|-88| = 88 unit ^{3}

Exercise 2.10

Given $\vec{A} = 6\vec{i} + 3\vec{j} + 2\vec{k}$ and $\vec{B} = 2\vec{i} + 2\vec{j} + \vec{k}$, find

(i) the unit vector in the direction of $\vec{B}$

(ii) the projection of $\vec{A}$ onto $\vec{B}$.

(iii) Determine whether $\vec{A}$ and $\vec{B}$ is parallel or not.

Answers:

(i) unit vector = $\frac{1}{3}\left(2\vec{i} + 2\vec{j} + \vec{k}\right)$

(ii) proj. of $\vec{A}$ onto $\vec{B}$ = $\frac{\left(\vec{A}\cdot\vec{B}\right)}{\left|\vec{B}\right|} = \frac{20}{3}$

(iii) Not Parallel

Exercise 2.11

Position vectors for points A, B, C and D are $\vec{a}$,$\vec{b}$, $3\vec{a} + 2\vec{b}$ and $2\vec{b} - \vec{a}$, respectively. Express the vector $2\overrightarrow{DB} - 3\overrightarrow{CA}$ in terms of $\vec{a}$ and $\vec{b}$.

Answers:

$2\overrightarrow{DB} - 3\overrightarrow{CA} = 8\vec{a} + 4\vec{b}$

Exercise 2.12

Given $\vec{A} = 2\vec{i} - \vec{j} + \vec{k}$, $\vec{B} = \vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{C} = 3\vec{i} + \alpha\vec{j} + 5\vec{k}$, find the values of :

(i) λ so that $\vec{A} + \lambda\vec{k}$ and $\vec{B} - \lambda\vec{i}$ are perpendicular.

(ii) α so that $\vec{A}, \vec{B}$ and $\vec{C}$ are coplanar.

Answers:

(i) $\lambda = -\frac{3}{4}$

(ii) $\alpha = 4$

Exercise 2.13

If $\vec{a} = 2\vec{i} - 3\vec{j} + 5\vec{k}$, $\vec{b} = 2\vec{i} - 4\vec{j} + 5\vec{k}$ and $\vec{c} = 12\vec{i} + \vec{j} - 4\vec{k}$, find :

(i) the unit vector in the direction of $\vec{a} + 2\vec{b} - \vec{c}$

(ii) the projection of $-3\vec{a} + \vec{c}$ on $\vec{a} + 2\vec{b} - \vec{c}$.

Answers:

(i) the unit vector = $\dfrac{\left(-6\vec{i} - 12\vec{j} + 19\vec{k}\right)}{\sqrt{541}}$

(ii) the projection = - 22.23

Exercise 2.14

Given three points A(2, 1, 3) , B(-1, 3, 4) and C(3, 1, 1), find :

(i) the angle between $\overrightarrow{CA}$ and the positive z-axis

(ii) the area of a triangle ABC

(iii) the equation of a plane having point C and perpendicular to $\overrightarrow{BC}$ and $\overrightarrow{BA}$

Answers:

(i) $\gamma = \cos^{-1}\left(\dfrac{2}{\sqrt{5}}\right) = 26.57°$

(ii) area of ABC = $\dfrac{1}{2}\sqrt{45}$ unit^2

(iii) eq.of the plane ➔ -4x –5y –2z + 19 = 0

Exercise 2.15

Given two position vectors, $\overrightarrow{OP} = 4\vec{i} - \vec{j} + 2\vec{k}$ and $\overrightarrow{OQ} = 3\vec{i} + 2\vec{j} + 3\vec{k}$, find a vector $\vec{u}$ which is parallel to $\overrightarrow{PQ}$.

Answers:

$\vec{u} = t\left(-\vec{i} + 3\vec{j} + \vec{k}\right)$

Exercise 2.16

Find parametric equations for the line through $P_o(2,3,0)$ and perpendicular to the vectors $\vec{u} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{v} = 3\vec{i} + 4\vec{j} + 5\vec{k}$.

Answers:

parametric eq. of L : x = -2t +2 ; y = 4t + 3 ; z = -2t

Exercise 2.17

A parametric equation of line L is $\begin{cases} x = -5 - 4t \\ y = 1 - t \\ z = 3 + 2t \end{cases}$, and an equation of plane S is $x + 2y + 3z - 9 = 0$.

(i) Determine whether the line L and the plane S are parallel.

(ii) Does the line L lies on the plane S ? Explain.

Answers:

(i) show : // iff n.v = 0

(ii) subs. (-5,1,3) in eq. if satisfies the eq. then it lies on the plane ; since 6 ≠ 9 ➔ doesn't lies on the plane.

Exercise 2.18

Given position vectors $\vec{P} = 8\vec{i} + 2\vec{j} - 3\vec{k}$, $\vec{Q} = 3\vec{i} - 6\vec{j} + 4\vec{k}$ and $\vec{R} = 2\vec{i} - 2\vec{j} - 2\vec{k}$.

Find a unit vector parallel to $\frac{1}{2}\vec{R} - 2\vec{Q} + \vec{P}$.

Answers:

$$\hat{u} = \frac{3}{\sqrt{322}}\vec{i} + \frac{13}{\sqrt{322}}\vec{j} - \frac{12}{\sqrt{322}}\vec{k}$$

Exercise 2.19

Given points A(2,-1,1), B(3,-4,5), C(2,1,-2) and D(1,2,-3).

(i) Find the area of the triangle with sides $\overrightarrow{AC}$ and $\overrightarrow{AD}$.

(ii) Show that position vectors of A, B and D are coplanar.

Answers:

(i) area = $\frac{1}{2}\left\|\overrightarrow{AC} \times \overrightarrow{AD}\right\| = \frac{\sqrt{14}}{2}$ unit^2

(ii) show : $\overrightarrow{OA} \bullet \left(\overrightarrow{OB} \times \overrightarrow{OD} \right) = 0 \sim$ coplanar.

Exercise 2.20

Given two vectors $\vec{a} = 2\vec{i} + t\vec{j} + (t-2)\vec{k}$ and $\vec{b} = t\vec{i} + 3\vec{j} + t\vec{k}$.

(i) Find t so that $\vec{a}$ is perpendicular to $\vec{b}$.

(ii) Find the projection of $\vec{a}$ on $\vec{b}$ when t = 1.

Answers:

(i) t = 0 , -3

(ii) $\left\| \text{proj.}\vec{v} \text{ onto } \vec{u} \right\| = \dfrac{\left| \vec{u} \bullet \vec{v} \right|}{\left\| v \right\|}$

$= \dfrac{4}{\sqrt{11}}$

Exercise 2.21

Suppose $\vec{A}$, $\vec{B}$, and $\vec{C}$ are unit vectors and mutually orthogonal. If $\vec{D} = 5\vec{A} - 6\vec{B} + 3\vec{C}$, find the magnitude of $\vec{D}$.

Answers:

$\left\| \vec{D} \right\| = \sqrt{70}$ unit; $\vec{D} = \vec{5} - 6\vec{j} + 3\vec{k}$

Exercise 2.22

U, V and W are points whose coordinates are (2,3,4) , (1,1,-8) and (5,6,-20) respectively, find

(i) A unit vector perpendicular to the plane UVW through the point U.

(ii) The equation of the plane UVW.

(iii) The area of UVW.

Answers:

(i) $\dfrac{1}{3\sqrt{1185}}\left(84\vec{i} - 60\vec{j} + 3\vec{k}\right)$

(ii) 84x – 60y + 3z = 0

(iii) 51.64 unit^2

Exercise 2.23

Given $\vec{A} = -4\vec{i} + 2\vec{j} - 5\vec{k}$, $\vec{B} = \frac{1}{2}\vec{i} + 6\vec{j} + 2\vec{k}$ and $\vec{C} = -2\vec{i} + 3\vec{j} - 5\vec{k}$.

(i) Determine whether vectors $\vec{A}$ and $\vec{B}$ are orthogonal?

(ii) Find the direction cosine of $\vec{C}$.

(iii) Find the projection of the vector $\vec{A}$ on the vector $\vec{C}$.

Answers:

(i) Since $\vec{A} . \vec{B} = 0$ then $\vec{A}$ and $\vec{B}$ are orthogonal.

(ii) $\cos\alpha = -\frac{2}{\sqrt{38}}$, $\cos\beta = \frac{3}{\sqrt{38}}$, $\cos\gamma = -\frac{5}{\sqrt{38}}$

(iii) $\vec{A} . \vec{C} = \frac{39}{\sqrt{38}}$

Exercise 2.24

(a) Given

$$\vec{u} = 4\vec{i} - (2 + x)\vec{j} + \vec{k}$$
$$\vec{v} = -(1 - x)\vec{i} + 3\vec{j} - 5\vec{k}$$
$$\vec{w} = 2\vec{i} + 3\vec{j} + (2x - 1)\vec{k}$$

Find x if $\vec{u} \cdot \vec{v} - 3(\vec{v} \cdot \vec{w}) = 0$

Answers:

$x = \frac{51}{25}$

Exercise 2.25

Show that the four points whose coordinates are A(1,0,0), B(2,1,0), C(3,2,1) and D(4,3,2) are coplanar.

Answers:

show $\overrightarrow{AB} \bullet \left(\overrightarrow{AC} \times \overrightarrow{AD}\right) = 0 \sim$ COPLANAR

Chapter Three

POWER SERIES

CHAPTER 3 : POWER SERIES

POWER SERIES:

A **power series** (in one variable) is an **infinite series** of the form f(x) = $\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$
where a_n represents the coefficient of the *n*th term, c is a constant, and x varies around c (the series as being *centered* at c).

Negative powers are not permitted in a power series, for instance $1 + x^{-1} + x^{-2} + \ldots$ is not considered a power series. Similarly, fractional powers such as $x^{1/2}$ are not permitted. The coefficients a_n are not allowed to depend on *x*, thus for instance: $e^x + e^x x + e^x x^2 + \ldots$ is not a power series.

Taylor and Maclaurin Series

Definition: If f has derivatives of all orders at x_0, then we call the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = \mathbf{f}(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + f'''(x_0)\frac{(x-x_0)^3}{3!} + \ldots + f^{(k)}(x_0)\frac{(x-x_0)^k}{k!} + \ldots \quad \textbf{where k = 0, 1, 2, 3 …}$$

*the **Taylor series** for f about* $x = x_0$

In the special case where $x_0 = 0$, this series becomes

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x)^k = \mathbf{f(0)} + f'(0)(x) + f''(0)\frac{(x)^2}{2!} + f'''(0)\frac{(x)^3}{3!} + \ldots + f^{(k)}(0)\frac{(x)^k}{k!} + \ldots$$

the **Maclaurin's series** for f

A. Taylor series

Example 1:

Find the Taylor series for $\frac{1}{x}$ about x = 1.

Solution:

Let $f(x) = \frac{1}{x}$, $f(1) = 1$

$f'(x) = -\frac{1}{x^2}$, $f'(1) = -1$

$f''(x) = \frac{2}{x^3}$, $f''(1) = 2$

$f'''(x) = \frac{-6}{x^4}$, $f'''(1) = -6$

Thus , the Taylor series for $\frac{1}{x}$ about x = 1 is

$$f(x) = \frac{1}{x} = 1 - (x-1) + 2\frac{(x-1)^2}{2!} + (-6)\frac{(x-1)^3}{3!} + \ldots$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \ldots \text{ (first 4 terms)}$$

Example 2:

Obtain the first four terms in the expansion of sin x as a power series of $(x - \frac{\pi}{2})$

Solution:

Let $f(x) = \sin x$, $f(\frac{\pi}{2}) = 1$

$f'(x) = \cos x$, $f'(\frac{\pi}{2}) = 0$

$f''(x) = -\sin x$, $f''(\frac{\pi}{2}) = -1$

$f'''(x) = -\cos x$, $f'''(\frac{\pi}{2}) = 0$

$f^{(4)}(x) = \sin x, \qquad f^{(4)}(\frac{\pi}{2}) = 1$

$f^{(5)}(x) = \cos x, \qquad f^{(5)}(\frac{\pi}{2}) = 0$

$f^{(6)}(x) = -\sin x, \qquad f^{(6)}(\frac{\pi}{2}) = -1$

$$\sin x = f(\frac{\pi}{2}) + f'(\frac{\pi}{2})(x-\frac{\pi}{2}) + f''(\frac{\pi}{2})\frac{(x-\frac{\pi}{2})^2}{2!} + f'''(\frac{\pi}{2})\frac{(x-\frac{\pi}{2})^3}{3!} + \ldots$$

$$= 1 - (x-\frac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^4}{4!} - \frac{(x-\frac{\pi}{2})^6}{6!} + \ldots$$

$$= 1 - (x-\frac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^4}{24} - \frac{(x-\frac{\pi}{2})^6}{720} + \ldots$$

Example 3:

Obtain the first four terms in the expansion of tan x as a power series of $(x + \frac{\pi}{4})$. :

Solution:

Let $f(x) = \tan x, \quad f(-\frac{\pi}{4}) = -1$

$f'(x) = \sec^2 x, \quad f'(-\frac{\pi}{4}) = 2$

$f''(x) = 2\sec x \sec x \tan x = 2\sec^2 x \tan x, \quad f''(-\frac{\pi}{4}) = -4$

$f'''(x) = 2\sec^2 x(\sec^2 x) + \tan x(4\sec x \sec x \tan x), \quad f'''(-\frac{\pi}{4}) = 16$

$$\tan x = -1 + 2(x+\frac{\pi}{4}) - 4\frac{(x+\frac{\pi}{4})^2}{2!} + 16\frac{(x+\frac{\pi}{4})^3}{3!} + \ldots$$

$$= -1 + 2(x+\frac{\pi}{4}) - 2(x+\frac{\pi}{4})^2 + \frac{8(x+\frac{\pi}{4})^3}{3} + \ldots$$

Example 4

Expand f(x) = $\frac{x}{1+x}$ as a series of ascending power of (x-1) up to the term containing $(x-1)^3$. use your expansion to estimate f(1.1) correct to four decimal places.

Solution:

Example 5

Obtain the Taylor's expansion of the function f(x) = x lnx as far as the term in $(x - 1)^4$.
Hence approximate f(1.5).

Solution:

Exercises:

1. Obtain the Taylor series expansion of $\frac{1}{1+x}$ about x = 2
2. Expand f(x) = $\frac{\ln x}{x}$ as a series of ascending powers of (x – 1) up to the term containing $(x - 1)^3$. Use your expansion to estimate f(1.1) correct to four decimal places.
3. Expand e^x in powers of (x – 1) up to the first five terms. Hence, estimate $e^{1.5}$ to four decimal places.

B. Maclaurin series

Example 1

Find the first four terms of the Maclaurin's series for a) e^x and $\ln(1+x)$ b) Hence deduce the expansion for e^{x^2} and $\ln(1-2x)$

Solution:

a) Let $f(x) = e^x$, $f(0) = 1$

$f'(x) = e^x$, $f'(0) = 1$

$f''(x) = e^x$, $f''(0) = 1$

$f'''(x) = e^x$, $f'''(0) = 1$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

let $f(x) = \ln(1+x)$, $f(0) = 0$

$f'(x) = \dfrac{1}{1+x} = (1+x)^{-1}$, $f'(0) = 1$

$f''(x) = -(1+x)^{-2}$, $f''(0) = -1$

$f'''(x) = 2(1+x)^{-3}$, $f'''(0) = 2$

$f^{(4)}(x) = -6(1+x)^{-4}$, $f^{(4)}(0) = -6$

$$\ln(1+x) = x - \frac{x^2}{2!} + 2\frac{x^3}{3!} - 6\frac{x^4}{4!} + \ldots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} +$$

b) $$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \ldots.$$

$$= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots.$$

$$\ln(1-2x) = -2x - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} +$$

$$= -2x - \frac{4x^2}{2} - \frac{8x^3}{3} - \frac{16x^4}{4} +$$

$$= -2x - \frac{4x^2}{2} - \frac{8x^3}{3} - 4x^4 + \ldots$$

Example 2

Find the Maclaurin series for sin x until the term x^5. Hence deduce the series for $\sin x^2$.
Using the series, estimate $\int_0^1 x \sin x^2 dx$

Solution:

Let f(x) = sin x, f(0) = 0

$f'(x) = \cos x$, $f'(0) = 1$

$f''(x) = -\sin x$, $f''(x) = 0$

$f'''(x) = -\cos x$, $f'''(0) = -1$

$f^4(x) = \sin x$, $f^4(0) = 0$

$f^5(x) = \cos x$, $f^5(0) = 1$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

Hence $\sin x^2 = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} + \ldots$

$$= x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} + \ldots$$

$$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \ldots$$

$$\int_0^1 x \sin x^2 dx = \int_0^1 x(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \ldots)dx$$

$$= \int x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} dx$$

$$= \left[\frac{\mathbf{x}^4}{4} - \frac{\mathbf{x}^8}{48} + \frac{\mathbf{x}^{12}}{1440} \right]_0^1 = 0.2299 \text{ (4 decimal places)}$$

Example 3

Find the first 4 terms of the Maclaurin series for ln(1- x). Hence deduce the series for ln(1 - $\frac{1}{2}x$). Find the value of $\ln\left(\frac{1}{2}\right)$ correct to four significant digits given $\ln\left(\frac{1-x}{1-\frac{1}{2}x}\right) = \ln\left(\frac{1}{2}\right)$

Solution:

Let f(x) = ln(1- x), f(0) = 0

$f'(x) = \frac{1}{1-x}(-1) = -(1-x)^{-1}$, $\quad f'(0) = -1$

$f''(x) = (1-x)^{-2}(-1)$, $\quad f''(0) = -1$

$f'''(x) = 2(1-x)^{-3}(-1)$, $\quad f'''(0) = -2$

$f^4(x) = 6(1-x)^{-4}(-1)$, $\quad f^4(0) = -6$

$$\ln(1-x) = -x - \frac{x^2}{2!} - 2\frac{x^3}{3!} - 6\frac{x^3}{3!} + \ldots$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

Thus, $\ln(1 - \frac{1}{2}x) = -\frac{1}{2}x - \frac{(\frac{1}{2}x)^2}{2} - \frac{(\frac{1}{2}x)^3}{3} - \frac{(\frac{1}{2}x)^4}{4} + \ldots$

$$= -\frac{1}{2}x - \frac{x^2}{8} - \frac{x^3}{24} - \frac{x^4}{64} + \ldots$$

$$\ln\left(\frac{1-x}{1-\frac{1}{2}x}\right) = \ln(1-x) - \ln(1-\frac{1}{2}x)$$

$$= \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - \left(-\frac{1}{2}x - \frac{x^2}{8} - \frac{x^3}{24} - \frac{x^4}{64} + \ldots\right)$$

$$= -\frac{1}{2}x - \frac{3}{8}x^2 - \frac{7}{24}x^3 - \frac{15}{64}x^4 + \ldots$$

When $\left(\dfrac{1-x}{1-\frac{1}{2}x}\right) = \left(\dfrac{1}{2}\right)$

$2- 2x = 1 - \frac{1}{2}x$

$-\frac{3}{2}x = -1$

$x = \frac{2}{3}$

$$\ln\left(\frac{1-x}{1-\frac{1}{2}x}\right) = \ln\left(\frac{1}{2}\right) = -\frac{1}{2}\left(\frac{2}{3}\right) - \frac{3}{8}\left(\frac{2}{3}\right)^2 - \frac{7}{24}\left(\frac{2}{3}\right)^3 - \frac{15}{64}\left(\frac{2}{3}\right)^4 + \ldots$$

$= - 0.6327$

Example 4

By using Maclaurin"s theorem. Obtain the first five terms of the function e^x . Hence , deduce the expansion of e^{-x^2} and evaluate $\int_0^1 e^{-x^2}\,dx$.

Solution:

Example 5

Obtain the Maclaurin's expansion of $(2 + x)^{-\frac{1}{2}}$ as far as the term x^3. Hence, estimate the value of $\int_0^1 \frac{1}{\sqrt{2+x}}dx$

Solution:

Exercises:

1. Obtain the Maclaurin expansion of xe^x to the powers of x^4.

2. Find the first four terms of the Maclaurin expansion for sin $(2x + \frac{\pi}{3})$. Hence, estimate sin 61^0 correct to four decimal places.

3. By using Maclaurin's theorem, obtain the expansion of ln(1+sinx) in ascending powers of x as far as the terms x^3. Given that sin(-x) = -sin x, deduce the expansion of ln(1 – sinx) until the terms x^3.Hence, find the expansion of $\ln\left(\frac{1-\sin x}{1+\sin x}\right)$ until the terms x^3.

Samples of Final Examination Questions:

OCT 2007

1. Obtain the expansion for sin x up to x^5 using Maclaurin's Series. Hence , using the expansion, deduce the expansion for $\sin(x^2)$ and evaluate $\int_0^1 \sin(x^2)dx$ correct to four decimal places.

ANSWER:

$$\textbf{Sin x} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...$$
$$\textbf{Sin x}^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + ...$$

$$\int_0^1 \sin(x^2)dx = \mathbf{0.3103}$$

OCT 2006

Expand f(x) = $\frac{\ln x}{x}$ as a series of ascending powers of (x – 1) up to the term containing $(x - 1)^3$. Use your expansion to estimate f(1.1) correct to four decimal places.

ANSWER:

$$\textbf{f(x) = (x – 1) -} \frac{3}{2}(x-1)^2 + \frac{11}{6}(x-1)^3 + ...$$
f(1.1) = 0.0868

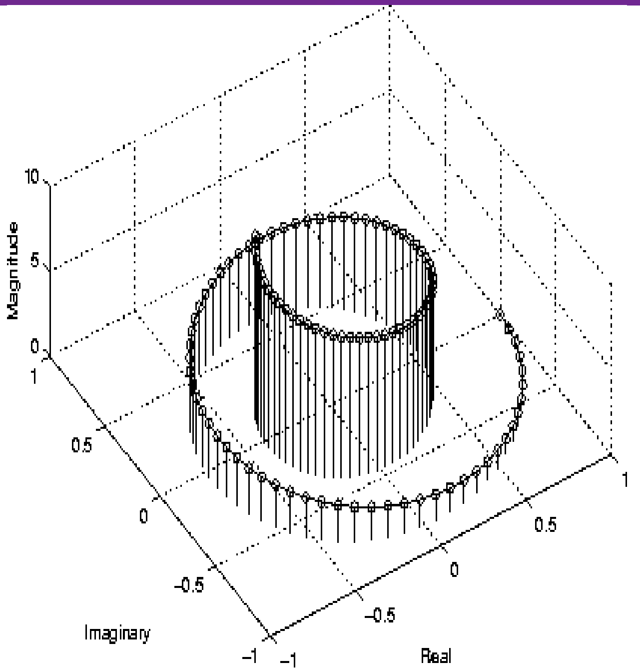
OCT 2000

QUESTION 1
By Taylor's expansion, obtain the first four terms of the expansion of sin x in terms of $\left(x - \frac{\pi}{4}\right)$.

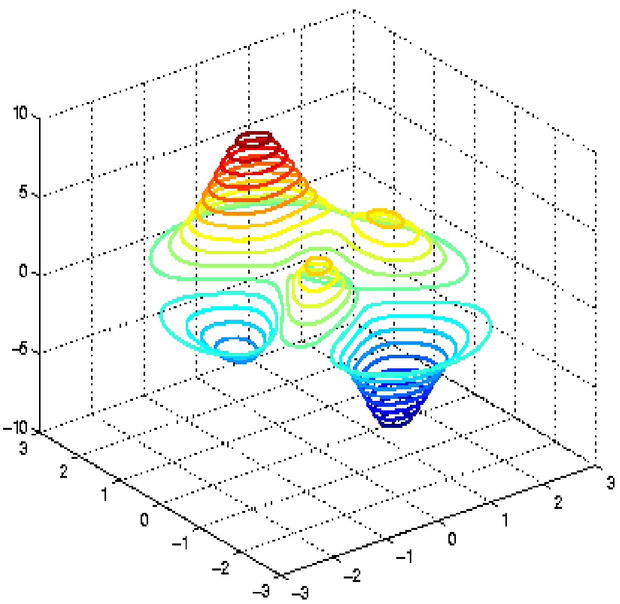
ANSWER:

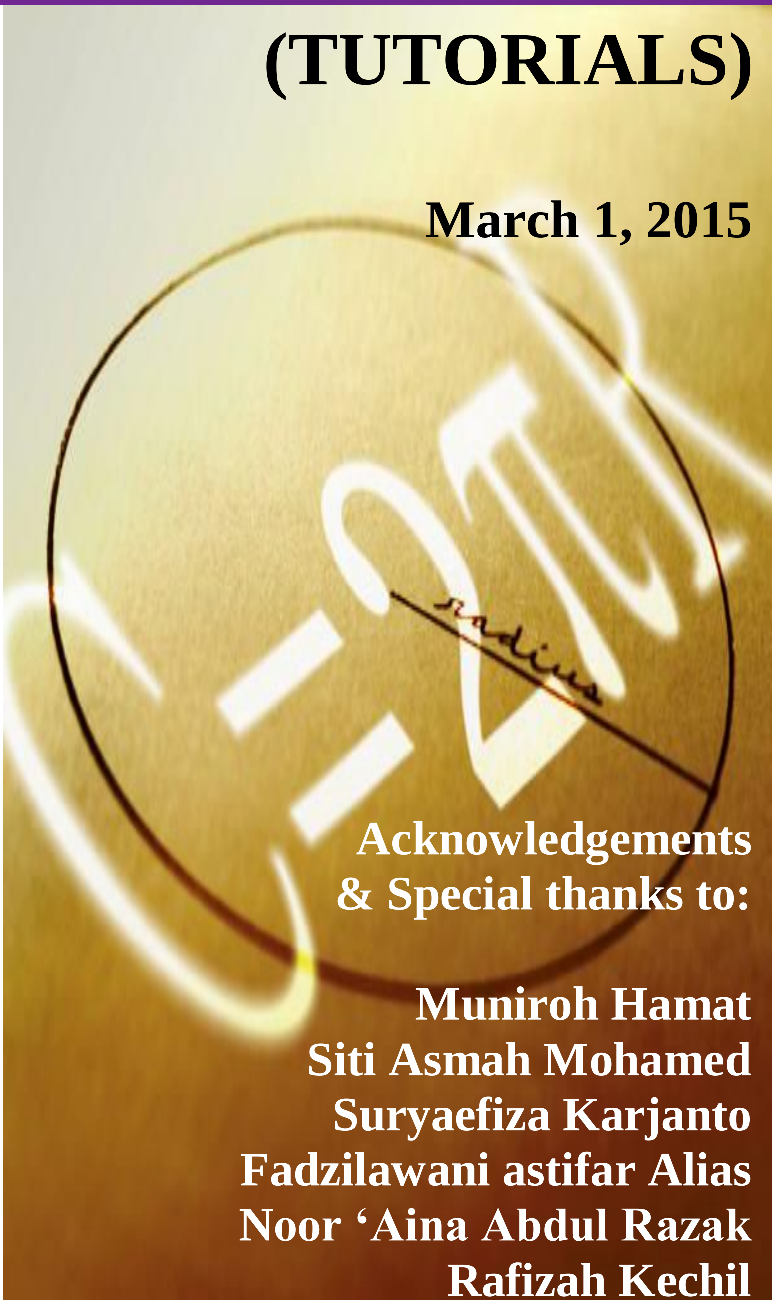
f(x) = sin x
= (1/√2) { (x- (π/4)) –1/2 (x- (π/4))^2 - 1/6 ((x- (π/4))^3 + …}

FURTHER MATHEMATICS FOR SCIENCE AND ENGINEERING



Twenty Contours of the peaks Function





(TUTORIALS)

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TUTORIAL

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1. If $M = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$, find M^2, M^3 and M^4.

2. If $P = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $Q = \begin{bmatrix} 3 & 5 \\ 0 & 6 \end{bmatrix}$

   i. Determine whether the multiplication of P and Q is commutative.
   ii. Compute $(P+Q)^2 - (P^2 + Q^2 + 2PQ)$

3. Let $A = \begin{bmatrix} 4 & -1 \\ 6 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 3 \\ 3 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & -1 \\ 6 & 9 \\ 2 & 3 \end{bmatrix}$, find

   a) CB b) $B^T C^T$ c) $CA + CB$

4. Let $A = \begin{bmatrix} x & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 2 & x \\ 0 & -1 \end{bmatrix}$

   Find x such that $\det(AB) = \det(BA)$.

5. If $\begin{vmatrix} 1 & 0 & x \\ x & 1 & 0 \\ 0 & x & 1 \end{vmatrix} = 28$. Using the properties of determinant or

   otherwise evaluate $\begin{vmatrix} 1 & x & x+1 \\ x+1 & 1 & x \\ x & x+1 & 1 \end{vmatrix}$

6. Prove that $\begin{vmatrix} 2a & 2b & b-c \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix} = -2(a-b)^2(a+b)$

7. Use the theorems of determinants to find the value of

$$\begin{vmatrix} k+3 & 3m & 2k \\ 5 & 2 & 4 \\ m & 4 & 3m \end{vmatrix} - \begin{vmatrix} k+3 & 5 & m \\ 3m & 2 & 4 \\ k-3 & -1 & 2m \end{vmatrix}$$

8. Using the properties of determinant, prove that

$$\begin{vmatrix} x+y & z & -x \\ x+z & y & -z \\ y+z & x & -y \end{vmatrix} = \begin{vmatrix} y & x & z \\ x & z & y \\ z & y & x \end{vmatrix}$$

9. If $\begin{vmatrix} p & q & r \\ s & t & u \\ v & w & x \end{vmatrix} = 5$, use the properties of determinants to find

$$\begin{vmatrix} 2s+p & 2t+q & 2u+r \\ -v & -w & -x \\ 2p & 2q & 2r \end{vmatrix}.$$

10. Assume that $\begin{vmatrix} p & q & r \\ s & t & u \\ v & w & x \end{vmatrix} = -4$. Using the properties of determinants show that

$$\begin{vmatrix} r+q & 2p & r \\ u+t & 2s & u \\ x+w & 2v & x \end{vmatrix} = 8.$$

Answer:

1) $M^2 = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$, $M^3 = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$, $M^4 = \begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix}$

2) i) $PQ = \begin{pmatrix} 3 & 23 \\ 6 & 34 \end{pmatrix}$,

$QP = \begin{pmatrix} 13 & 29 \\ 12 & 24 \end{pmatrix}$

$PQ \neq QP \therefore$ Not commutative

ii) $\begin{pmatrix} 10 & 6 \\ 6 & -10 \end{pmatrix}$

3) $CB = \begin{pmatrix} -3 & 14 \\ 27 & 0 \\ 9 & 0 \end{pmatrix}$, $B^T C^T = \begin{pmatrix} -3 & 27 & 9 \\ 14 & 0 & 0 \end{pmatrix}$, $CA + CB = \begin{pmatrix} 7 & 1 \\ 105 & 75 \\ 35 & 25 \end{pmatrix}$

4) $x = 2$

5) 56

6) proven

7) 0

8) proven

9) -20

10) Shown

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1. Find matrix B if $A=\begin{pmatrix}1 & 6\\ -3 & 5\end{pmatrix}$ and matrix $AB=\begin{pmatrix}16 & -6\\ -2 & -5\end{pmatrix}$.

2. Given matrix $P^{-1}Q^{-1}=\begin{bmatrix}3 & 5\\ 2 & 4\end{bmatrix}$.Hence find QP.

3. By using elementary row transformation, find inverse of the matrix

$$B=\begin{pmatrix}1 & -1 & 0\\ -1 & 0 & 1\\ 0 & 2 & -1\end{pmatrix}$$

4. Given $A=\begin{bmatrix}1 & -1 & 3\\ 2 & 1 & 0\\ 1 & 2 & 2\end{bmatrix}$. Find A^{-1} using
   i) the elementary row operation
   ii) the adjoint method.

5. If $A=\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & -1\\ -1 & 1 & 0\end{bmatrix}$ find $\operatorname{adj}(A)$. Hence, verify that the inverse of A is $\begin{bmatrix}1/3 & 0 & -2/3\\ 1/3 & 0 & 1/3\\ 1/3 & -1 & 1/3\end{bmatrix}$.

   Use the result to find the matrix Y in the following equation

$$3\begin{bmatrix}-3 & 2 & -1\\ 5 & 1 & 4\end{bmatrix}-2Y\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & -1\\ -1 & 1 & 0\end{bmatrix}=\begin{bmatrix}-6 & 2 & 10\\ 7 & 4 & -2\end{bmatrix}$$

6. Consider the following linear equations.

$$2x = -12 + 5y - z$$
$$2y = 6 - x - 3z$$
$$2z = -6 - 2x - 6y$$

i) Find $|A|$ using cofactor (Laplace) expansion.

ii) Find the solutions using adjoint method.

7. Given $A = \begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix}, B = \begin{pmatrix} -10 & -2 & -1 \\ 10 & 5 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 4 \\ 0 & 0 \\ 2 & -1 \end{pmatrix}$. Find

(i) A^{-1} using the elementary row operation method,

(ii) matrix D such that $D = 2A^{-1} + BC - I^3$.

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Answer:

1) $B = \begin{pmatrix} 4 & 0 \\ 2 & -1 \end{pmatrix}$

2) $QP = \begin{pmatrix} 2 & -5/2 \\ -1 & 3/2 \end{pmatrix}$

3) $B^{-1} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}$

4) $A^{-1} = \begin{pmatrix} 2/15 & 8/15 & -1/5 \\ -4/15 & -1/15 & 2/5 \\ 1/5 & -1/5 & 1/5 \end{pmatrix}$

5) $\text{Adj }(A) = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & -3 & 1 \end{pmatrix}$

$Y = \begin{pmatrix} -2 & 13/2 & -1/2 \\ 7/2 & -7 & -1/2 \end{pmatrix}$

6) $|A| = -46$, $x = -\dfrac{183}{23}$, $y = \dfrac{3}{23}$, $z = \dfrac{105}{23}$

7) i) $A^{-1} = \begin{pmatrix} -7 & 2 \\ 4 & -1 \end{pmatrix}$, ii) $D = \begin{pmatrix} -27 & -35 \\ 24 & 34 \end{pmatrix}$

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1. Given $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$. Find A^{-1} using

   i) the elementary row operation
   ii) the adjoint method.

   From Tutorial 2

   Hence, solve the following equations:

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 1 \\ 2x_1 + x_2 \quad &= 3 \\ -x_1 - 2x_2 - 2x_3 &= 2 \end{aligned}$$

2. By using the answer from Q1, solve the following equations

$$\begin{aligned} x-y+3z&=1 \\ 2x+y&=3 \\ x+2y+2z&=2 \end{aligned}$$

3. Consider the following system of linear equation below:

$$\begin{aligned} 2x + 5y + z &= -12 \\ 2x + 2y + z &= 6 \\ 2x + 6y &= -6 \end{aligned}$$

   Find the inverse of the coefficient matrix by using adjoint method. Hence, apply matrix inversion method to solve for x, y and z.

4. Given $M = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} -8 & 9 & 12 \\ -2 & 1 & -1 \\ 7 & -3 & -6 \end{pmatrix}$

   i) Calculate $P = M + N - 5I$ where I is identity matrix.
   ii) Find MP. Hence, deduce the matrix M^{-1}.
   iii Determine N^{-1} by using adjoint method. Hence, solve the following system of linear equations

$$\begin{aligned} -8x + 9y + 12z &= 77 \\ -2x + y - z &= 4 \\ 7x - 3y - 6z &= -19 \end{aligned}$$

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5. Using the adjoint method, find the inverse of matrix $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$.

Hence, solve the following system of linear equations.

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 3z &= -1 \\ x + 4y + 9z &= -9 \end{aligned}$$

6. Find the inverse of matrix A where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

using elementary row operations. Hence, solve the given system of linear equations by matrix inversion method.

$$\begin{aligned} x + 2y + 3z &= 5 \\ 2x + 5y + 3z &= 3 \\ x \qquad + 8z &= 17 \end{aligned}$$

7. Given
$$\begin{aligned} y &= 2x + b(x - a) \\ z &= x + b(y - a) + 2y \\ y &= 2z + b(z - a) \end{aligned}$$
Show that $y = \dfrac{ab}{2 + b}$ by using Cramer's rule. a and b are constants.

8. Given the following system of equation, solve for z using Cramer's rule.

$$\begin{aligned} -2x + 3y - z &= 1 \\ x + 2y - z &= 4 \\ -2x - y + z &= -3 \end{aligned}$$

9. Solve the following system of equations using Cramer's Rule
$$\begin{aligned} 3x - y + 4z + 2 &= 0 \\ x + 2y - z + 3 &= 0 \\ -2x + 3y + z - 5 &= 0 \end{aligned}$$

10. Show that the coefficient matrix A of the following system of equation is a non-singular matrix.

$$\begin{aligned} 3x + 2y - z &= 1 \\ 5x + 3y + 2z &= p \\ -2x + y - 3z &= -1 \end{aligned}$$
Hence, find the value of p using the Cramer's rule if $y = 3$.

11. Find the inverse of matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ using the adjoint method.

Hence solve the following system of linear equations.

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 3z &= -1 \\ x + 4y + 9z &= -9 \end{aligned}$$

12. Using Cramer's rule, find x satisfying the following system of equations.

$$\begin{bmatrix} \lambda & 1 & 0 \\ 1 & \lambda & 1 \\ 0 & 1 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Answer:

1) $A^{-1} = \begin{pmatrix} 2/15 & 8/15 & -1/5 \\ -4/15 & -1/15 & 2/5 \\ 1/15 & -1/5 & 1/5 \end{pmatrix}$, $x = \frac{32}{15}$, $y = -\frac{19}{15}$, $z = -\frac{4}{5}$

2) $x = \frac{4}{3}$, $y = \frac{1}{3}$, $z = 0$

3) $x = 2, y = -12, z = 0$

4)) $P = \begin{pmatrix} -11 & 9 & 15 \\ -1 & 1 & 1 \\ 8 & -6 & -10 \end{pmatrix}$, $MP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $M^{-1} = \begin{pmatrix} -11/2 & 9/2 & 15/2 \\ -1/2 & 1/2 & 1/2 \\ 4 & -3 & -5 \end{pmatrix}$,

$N^{-1} = -\frac{1}{111}\begin{pmatrix} -9 & 18 & -21 \\ -19 & -36 & -32 \\ -1 & 39 & 10 \end{pmatrix}$

$x = 2$, $y = 9$, $z = 1$

5) $x = 1$, $y = 2$, $z = -2$

6) $A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$, $x = 1$, $y = -1$, $z = 2$

7) Shown

8) $x = 2, y = 3, z = 4$

9) $x = -2, y = 0, z = 1$

10) $p = 8$

11) $x = 1, y = 2, z = -2$

12) $x = \frac{\lambda^2 - 2}{\lambda^3 - 2\lambda}$

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1. Solve the following system of linear equations by using Gauss elimination method

$$x+y-4z=-30$$
$$3x+2y-z=-15$$
$$5x+3y+2z=0$$
$$3x+y+3z=11$$

2. Determine the value of m such that the non-homogeneous system

$$x_1+x_2+2x_3=2$$
$$2x_1+3x_2+2x_3=3$$
$$2x_1+3x_2+(m^2-7)x_3=m$$

has i) no solution
ii) infinitely many solution
iii) exactly one solution

3. Consider the non-homogeneus system of equations:

$$x=4-3y+z+2w$$
$$2x=2-4y+2z$$
$$-2x=10-2z+8w$$
$$3x=-3-4y+3z-4w$$

Use Gauss elimination method to determine whether
i) the system is linearly dependent or independent
ii) the system is consistent or inconsistent. If consistent, find the solution

4. A system in the form of $[A|B]$ is given as

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & (b^2-14) & b+2 \end{array}\right]$$

Where b is a real number. Determine value(s) of b so that

i) The rank of A is equal to 3
ii) The system is dependent
iii) The system has infinitely many solutions
iv) The system has no solution

5. Consider the non-homogeneous system of equations

$$x+3y+2z=3$$
$$3x+7y+5z=5$$
$$2x+4y+3z=k$$

Use Gauss Elimination method to show that there is only one value of k for the system to be consistent. Hence solve the equations using this value of k.

6. Consider the following system of non-homogeneous equations $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ -2 & 0 & 0 & 1 \\ 1 & 2 & 2 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 4 \end{bmatrix}$$

i) Determine r(A) and r(A l B).
ii) State the type of the solution and hence, give the solution.
iii) State the dependency of the system.

7. Consider the non homogeneous system below

$$2x+4y+z=-2$$
$$3x+6y+2z=3$$
$$2x+y+2z=1$$

i) write the augmented matrix of the system.
ii) Determine whether the system is linearly dependent or linearly independent.
iv) Is the system consistent or inconsistent?.If consistent, give the solution by using backward.

8. Using Gaussian elimination, determine the value(s) of k so that the system of equations

$$x+2y+z=3$$
$$2x+2y-2z=4$$
$$x+y+(k^2-5)z=k$$

has i) no solution,
ii) infinitely many solutions,
iii) a unique solution.

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9. Consider the following non-homogeneous system.

$$x + y + z + w = 2$$
$$x + 3y + 3z = 2$$
$$-x - y - 2z + w = 3$$

i) Find the rank of the coefficient matrix and augmented matrix.
ii) Is the system of equations linearly dependent or linearly independent? Why?
iii) Determine whether the system of equations is consistent or inconsistent.
iv) Why? If consistent find the solutions.

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Answer :

1) $x = -4,\ y = 2,\ z = 7$

2) i) $m = -3$ ii) $m = 3$ iii) $m \neq \pm 3$

3) $x = -5 + s - 4t,\ y = 3 + 2t,\ z = s,\ w = t$

4) i) $b \neq \pm 4$, ii) $b = 4$, iii) $b = 4$, iv) $b = -4$

5) $k = 2,\ x = -3 - \tfrac{1}{2}s,\ y = 2 - \tfrac{1}{2}s,\ z = s$

6) i) r(A) = 3, r(A l B) = 3
ii) many solution,
x = 6-t , y = t, z = -3, w = -2
iii) linearly dependent

7) $x = -13,\quad y = 3,\quad z = 12$

8) i) $k = -2$
ii) $k = 2$
iii) $k \neq \pm 2$

9) $x = 2 - \tfrac{3}{2}s,\quad y = 5 - \tfrac{3}{2}s,\quad z = -5 + 2s,\quad w = s$

1. Find the values of p for which the homogeneous system

$$\begin{aligned} 2x - y + pz &= 0 \\ px + y + z &= 0 \\ x - y - pz &= 0 \end{aligned}$$

has non-trivial solution. Find the complete solution when $p = 1$.

2. Determine the value of k for the system of equations to have non-trivial solution

$$\begin{aligned} x + y - z &= 0 \\ kx - 4y + 3z &= 0 \\ -5x + 13y - 10z &= 0 \end{aligned}$$

3. Consider the following homogeneous equations

$$\begin{aligned} x+ky+2z&=0 \\ kx-y-z&=0 \\ 9x-2y-2z&=0 \end{aligned}$$

Determine the values of k so that the system will have a non-trivial solution.

4. Consider the following homogeneous system

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ 2x_1 + 3x_2 + kx_3 &= 0 \\ x_1 + kx_2 + 3x_3 &= 0 \end{aligned}$$

Using gauss elimination, determine the value(s) of k so that the system will have non- trivial solutions.

Answer :

1) $p = -1.78, \quad p = 0.28$
When $p = 1$, $x = 0, \quad y = 0, \quad z = 0$

2) $k = 2$

3) $k = \frac{9}{2}, \quad k = 2$

4) $k = -3, \quad k = 2$

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1. Given the matrix $B=\begin{pmatrix} 1 & 0 & 0 \\ -1 & -2 & -1 \\ 2 & 3 & 2 \end{pmatrix}$. Find the eigenvalues and the normal eigenvector corresponding to the smallest eigenvalue of B.

2. Given that $M=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ n^3 & -3n^2 & 3n \end{bmatrix}$, show that the characteristic equation of M is $(n-k)^3=0$. Find the eigenvalue and normal eigenvector when $n=-1$.

3. Let $A=\begin{bmatrix} -1 & 4 & 2 \\ 0 & 1 & 0 \\ -4 & 8 & 5 \end{bmatrix}$. Find
   i. The characteristic polynomial of matrix A
   ii. All eigenvalues of A and the corresponding eigenvector to the repeated eigenvalue.

4. Given $k=-1$ is an eigenvalue of matrix

$$A=\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \text{ find}$$

   i) Other eigenvalues.
   ii) Eigenvector corresponding to the largest eigenvalue.

5. Given matrix $A=\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$. Show that the eigenvector of A are $V_1=(-2t \quad t)^T$ and $V_2=(t \quad t)^T$. Hence, find the normal eigenvector corresponding to v_1.

6. Given $A=\begin{bmatrix} 7 & 0 & 5 \\ 0 & 5 & 0 \\ -4 & 0 & -2 \end{bmatrix}$

   Find the eigenvalues and the normal eigenvector corresponding to the smallest eigenvalue of A.

7. Given the following matrix $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$,

   i) find all the eigenvalues for the above matrix.
   ii) find the eigenvector corresponding to the largest eigenvalue found in (i).

8. Determine all the eigenvalues for the matrix

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

and the normal eigenvector corresponding to the smallest eigenvalue.

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Answer :

1) $k=1, \quad k=-1, \quad X=\begin{pmatrix}0\\-t\\t\end{pmatrix}, \quad U=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}$

2) $k=-1 \quad U=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}$

3) $k=1, \quad k=3, \quad X=s\begin{pmatrix}2\\1\\0\end{pmatrix}+t\begin{pmatrix}1\\0\\1\end{pmatrix}$

4) $k=2, \quad X=s\begin{pmatrix}1\\0\\1\end{pmatrix}+t\begin{pmatrix}-1\\1\\0\end{pmatrix}$

5) $k=1, \quad k=4$

6) $k=2, \quad k=3, \quad k=5 \quad U=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}$

7) $k=2, \quad k=4 \quad X=s\begin{pmatrix}1\\0\\0\end{pmatrix}+t\begin{pmatrix}0\\1\\1\end{pmatrix}$

8) $k=1, \quad k=2 \quad U=\frac{1}{\sqrt{6}}\begin{pmatrix}-2\\1\\1\end{pmatrix}$

1. Given the points $0(0,0,0)$, $A(2,-1,2)$, $B(4,-2,-5)$ and $C(-6,1,4)$. Find
   a) the vector projection of $\vec{BA}$ onto $\vec{OA}$
   b) the angle between $\vec{OA}$ and $2\vec{OB}+\vec{OC}$
   c) the area of the triangle ABC
   d) the volume of the parallelepiped formed by the vectors $\vec{OA}, \vec{OB}$ and $\vec{OC}$.
   e) The equation of the plane passing through the points A, B and C.

2. If $\vec{P} = 10\vec{i} - 3\vec{j} + 5\vec{k}$ and $\vec{Q} = 2\vec{i} + 6\vec{j} - 3\vec{k}$, find
   i) the unit vector perpendicular to both $\vec{P}$ and $\vec{Q}$
   ii) the vector projection of $\vec{P}$ onto $\vec{Q}$
   iii) determine whether $\vec{P}$ and $\vec{Q}$ are parallel or not.

3. If $\vec{M} = 6\vec{i} - 4\vec{j} + \vec{k}$ and $\vec{N} = 8\vec{i} + 5\vec{j} - 3\vec{k}$, find the direction cosines and the direction angles of the vector $\vec{M} X \vec{N}$.

4 For the vectors $\vec{a} = \vec{i} + 5\vec{j} - 2\vec{k}$ and $\vec{b} = 5\vec{j} - 3\vec{k}$, determine
   i. Vector projection of $\vec{a}$ along $\vec{b}$.
   ii. Angle between vectors $\vec{a}$ and $\vec{b}$.
   iii. The area of a triangle spanned by vectors $\vec{a}$ and $\vec{b}$.

5. Find the angles which the line joining the point $(1,-3,2)$ and $(3,-5,1)$ makes with the coordinate axes.

6. Consider the parallelepiped with sides $\vec{U} = 3\vec{i} + 2\vec{j} + \vec{k}$, $\vec{V} = \vec{i} + \vec{j} + 2\vec{k}$ and $\vec{W} = \vec{i} + 3\vec{j} + 3\vec{k}$. Find
   i. The area of the face determined by $\vec{U}$ and $\vec{W}$.
   ii. The acute angle between $\vec{V}$ and $\vec{U} \times \vec{W}$

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7. The position vector of $\vec{OP}$, $\vec{OQ}$ and $\vec{OR}$ is $\langle 0,0,0\rangle, \langle 1,2,3\rangle$, and $\langle 2,1,-1\rangle$ respectively. Find

   i) the area of the triangle PQR.

   ii) the vector projection of $\vec{QR}$ onto $\vec{OR}$.

   iii) the direction angles of vector $\vec{QR}$.

8. $P(2,4,-2), Q(-1,2,-8), R(0,-5,2)$ and $S(1,3,-1)$ are points in space.

   a) Show that P, Q and R are the vertices of a right triangle.

   b) Find the area of triangle PQR.

   c) Find the vector projection of $\vec{PR}$ onto $\vec{PS}$.

   d) Find the volume of a parallelepiped spanned by the vectors $\vec{PQ}$, $\vec{PR}$ and $\vec{PS}$.

   e) Find the equation of plane L containing points P, Q and R.

   f) Find the intersection point between plane L and line : $x-2=y=z-1$.

   g) Find the symmetrical equation of a line passing through point S and parallel to $\vec{V}=3\vec{i}-\vec{j}+2\vec{k}$

9. If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors and given

$$\vec{R}=(x+4y)\vec{a}+(2x+y+1)\vec{b} \quad ; \quad \vec{S}=(y-2x+2)\vec{a}+(2x-3y-1)\vec{b}$$

Find the values of x and y such that $3\vec{R}=2\vec{S}$.

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10. Let $P(1,4,-2)$, $Q(4,7,8)$ and $R(6,10,20)$ be three vertices in space

    i) Find a vector $\vec{V}$ perpendicular to the plane containing the vertices P, Q and R.
    ii) Find the area of the parallelogram with a vertex P and the vertices at the other ends of the sides adjoining this vertex are Q and R.
    iii) Find the direction angels of vector $\vec{V}$ along x-axis and z-axis

11. If $\vec{a} = 2\vec{i} - 2\vec{j} + \vec{k}$ and $\vec{b} = \vec{i} + 5\vec{j} - \vec{k}$, find the direction cosines of the vectors $\vec{a} \times \vec{b}$.

12. Let P(2,-1,1), Q(-3,2,0) and R(4,-5,3) be three point in space. Find
    (a) all direction angles of the vector $\vec{PR}$.
    (b) a unit vector perpendicular to both $\vec{PR}$ and $\vec{PQ}$.
    (c) the area of triangle PQR.
    (d) the equation of a plane passing through point P and normal to the vector $2i + 8j + 14k$.
    (e) If $M(4, \lambda, -3)$ is another point in space, find
        i. the value(s) of λ so that all vectors $\vec{PQ}$, $\vec{PR}$ and $\vec{PM}$ lie on the same plane (coplanar).
        ii. the volume of parallelepiped spanned by vectors $\vec{PQ}$, $\vec{PR}$ and $\vec{PM}$ when $\lambda = -1$.

13. Given three vectors
    $\vec{A} = -i + j - k$; $\vec{B} = 2i + j - 4k$ and $\vec{C} = -4i - 2j - tk$
    Calculate:
    i) the value of t such that vectors $\vec{B}$ and $\vec{C}$ are parallel.
    ii) the value of t such that vectors $\vec{B}$ and $\vec{C}$ are orthogonal.
    iii) $\vec{A} \bullet \left(\vec{B} \times \vec{C} \right)$ for $t = -1$.

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Answer :

1) a) $\langle 2, -1, 2 \rangle$

b) $103.77°$

c) 26.76 unit^2

d) 36 unit^3

e) $12x+52y-4z = -36$

2) i) $-\frac{21}{\sqrt{6397}}\vec{i}+\frac{40}{\sqrt{6397}}\vec{j}+\frac{66}{\sqrt{6397}}\vec{k}$

ii) $-\frac{13}{49}\langle 2, 6, -3 \rangle$

iii) not parallel

3) $\alpha = 84.06°, \quad \beta = 67.38°, \quad \delta = 23.47°$

4) i) $\frac{31}{34}\langle 0, 5, -3 \rangle$

ii) $\theta = 13.92°$

iii) 3.84 unit^2

5) $\angle A = 48.2°, \quad \angle B = 131.8°, \quad \angle C = 109.5°$

6) i) 11.05unit^2 ii) $70.57°$

7) i) 4.55 unit^2,

ii) $\frac{5}{6}\langle 2,1,-1 \rangle$

iii) $\alpha = 76.37°, \beta = 103.63°, \quad \delta = 160.52°$

8) a) Shown

b) 35.17 unit^2

c) $5\langle -1,-1,1 \rangle$

d) 61 unit^3

e) $-62x+24y+23z = -74$

f) $t = -\frac{9}{5}, \quad x = \frac{1}{5}, \; y = -\frac{9}{5}, \quad z = -\frac{4}{5}$

g) $\frac{x-1}{3} = \frac{y-3}{-1} = \frac{z+1}{2}$

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9. $x = 2$, $y = -1$

10. i) $\langle 6, -16, 3 \rangle$

ii) 8.675 unit^2

iii) $\alpha = 69.77^o$, $\gamma = 80^o$

11. $\cos\alpha = -\dfrac{3}{\sqrt{162}}$, $\cos\beta = \dfrac{3}{\sqrt{162}}$, $\cos\gamma = \dfrac{12}{\sqrt{162}}$

12. a) $\alpha = 65.9^o$, $\beta = 144.7^o$, $\gamma = 65.9^o$

b) $\dfrac{1}{\sqrt{264}}\langle -2,-8,-14 \rangle$

c) $\dfrac{1}{2}\sqrt{264}$ unit^2

d) $2x + 8y + 14z = 10$

e) i. $\lambda = \dfrac{11}{2}$

ii. 52 unit^2

13. i) $t = -8$

ii) $t = \dfrac{5}{2}$

iii) 21

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1. Given the lines in space

L_1 : $x = 2t + 1$, $y = 3t + 2$ $z = 4t + 3$

L_2 : $x = s + 2$, $y = 2s + 4$ $z = -4s - 1$

a) Find the parametric equation of the line through the point $P(3.-2,1)$ parallel to the line L_1.
b) The point in which the line L_1 meets the xy-plane.
c) The point of intersection of L_1 and L_2.

2. Let $\vec{P} = \vec{i} + 2\vec{j} - \vec{k}$, $\vec{Q} = -\vec{i} + \vec{j} + \vec{k}$ and $\vec{R} = -\frac{\pi}{2}\vec{i} - \pi\vec{j} + \frac{\pi}{2}\vec{k}$

i) Determine which two vectors are perpendicular to each other.
ii) Determine which two vectors are parallel to each other.
Give the reason to your answers.

3. A, B and C are points $(2,1,3)$, $(4,-1,-1)$ and $(-1,3,-6)$ respectively. Find the point D if $ABCD$ is a parallelogram with $\vec{AC}$ and $\vec{BD}$ as the diagonals. Determine whether $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ are coplanar?

4. i) Find the parametric equation of the line that passes through $(1,-2,3)$ and is parallel to $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$.
ii) Find the coordinates of the point where the line through $P(3,4,1)$ and $Q(5,1,6)$ crosses the xy - plane.
iii) Find the equation of the plane that passes through $D(0,1,1)$, $E(2,1,0)$ and $F(-2,0,3)$

5. Given $P(1,2,3)$, $Q(2,-1,5)$ and $R(-1,1,4)$, find
i) the area of the triangle PQR.
ii) the equation of that passes through the midpoint of PQ and perpendicular to QR

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6. Given

$$L_1: (x-2)\vec{i}+(y-1)\vec{j}+(z+3)\vec{k}=t(2\vec{i}-\vec{j}-\vec{k})$$

$$L_2:(x+1)\vec{i}+(y+2)\vec{j}+(z-6)\vec{k}=s(-3\vec{i}-3\vec{j}+9\vec{k})$$

i) Show that $A(2,1,-3)$ is the intersection point between lines L_1 and L_2
ii) Find the angle of intersection between by lines L_1 and L_2
iii) Find the equation of the plane determined by lines L_1 and L_2.

7. Given line $L: x = 2t+1$, $y = 5t-2$, $z = -t+3$ is perpendicular to plane S.

   i. Find the equation of plane S which consists point $(1,-1,0)$.
   ii. Determine the intersection point between L and S.

8. Let A , B and C be the position vectors of $A(3,-1,1)$, $B(4,1,4)$ and $C(6,0,4)$ respectively. Find

   i. a unit vector perpendicular to AB and AC
   ii. the vector projection of C onto AB
   iii. the volume of the parallelepiped spanned by A, B and C.
   iv. the parametric equation of the line passing through points A and B.

9. Given the equation of two lines

$$L_1: x=3+t, \quad y=1-2t, \quad z=3+3t$$

$$L_2: x=4+2s, \quad y=6+3s, \quad z=1+s$$

i) Find the point of intersection between L_1 and L_2.
ii) Find the equation of the plane determined by L_1 and L_2.

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10. Given
$$L_1 : (x-2)\vec{i} + (y-1)\vec{j} + (z+3)\vec{k} = t(2\vec{i} - \vec{j} - \vec{k})$$
$$L_2 : (x+1)\vec{i} + (y+2)\vec{j} + (z-6)\vec{k} = s(-3\vec{i} - 3\vec{j} + 9\vec{k})$$

i) Show that $M(2,1,-3)$ is the intersection point between lines L_1 and L_2.

ii) Find the angle of intersection between L_1 and L_2.

iii) Find the equation of the plane determined by lines L_1 and L_2.

11. Let $A(3,8,4)$ and $B(2,9,1)$ be two points in space. Find

i) the parametric equation of line that passes through the points A and B.

ii) the intersection point between the line obtain in i) and the zy-plane.

12. Let $Q(1,1,1)$ be a point, L be the line with the parametric equations $x = -1 + 3t$, $y = 2 + 2t$, $z = 5 + t$ and S be the plane with the equation $5x - 3y + 4z = 8$

i) Find an equation of a plane through Q and parallel to the plane S

ii) Find the intersection point between line L and plane S

13. The triangle ABC has its vertices at the points $A(-1,0,3), B(2,-1,-3)$A(-1,0,3) and C(5,1,-5). Find

i) $\vec{AB}$ and $\vec{AC}$.

ii) the area of triangle ABC.

iii) the equation of the plane containing the points A, B and C.

14. Given the vectors $\vec{a} = 3\vec{i} + \vec{j} + \vec{k}$, $\vec{b} = \vec{i} - 2\vec{j} + 5\vec{k}$ and $\vec{c} = 3\vec{i} + \lambda\vec{j} + 4\vec{k}$,

(i) state the condition for the three vectors to be coplanar. Hence find the value of λ so that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

(ii) find the volume of parallelepiped spanned by vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

15. Given four points $A(-2,0,3)$, $B(1,2,-1)$, $C(-2,-13/5, 26/5)$ and $D(16/5, -13/5, 0)$

Find the equation of the plane passing through points A and B that is parallel to $\vec{CD}$.

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16. Given the equation of two lines in space

$$L_1 : x = 2 + 3t\ ; \quad y = -4 - 2t\ ; \quad z = -1 + 4t$$
$$L_2 : x = 6 + 4s\ ; \quad y = -2 + 2s\ ; \quad z = -3 - 2s$$

(a) Find the parametric equation of the line through the point A(3,-1,0) and parallel to the line L_1.
(b) Find the intersection point of lines L_1 and L_2.
(c) Show that L_1 and L_2 are orthogonal to each other.

17. Given vectors

$$\vec{R} = 3i - j - 4k\ ; \quad \vec{s} = 2i + 5j - 2k \quad \text{and} \quad \vec{T} = -i + 6k$$

Find:

(i) a vector $\vec{M}$ having the opposite direction and twice the magnitude of vector $3\vec{R} - 2\vec{S}$.
(ii) the direction of vector $\vec{M}$.
(iii) the vector projection of vector $\vec{R}$ onto $\vec{S}$.
(iv) the angle between vector $\vec{S}$ and vector $\vec{T}$.

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Answer :

1. a) x =3 + 2t, y = -2 + 3t, z = -4t – 1
   b) $\left(-\frac{1}{2},-\frac{1}{4},0\right)$
   c) (1, 2, 3)

2. i) $\vec{P}\bullet\vec{Q}=0\Rightarrow\vec{P}$ perpendicular to $\vec{Q}$
   ii) $\vec{P}\times\vec{R}=\vec{0}\Rightarrow\vec{P}$ parallel to $\vec{R}$

3. $D(-3,5,-2)$, $\overrightarrow{AB}\bullet\left(\overrightarrow{AD}\times\overrightarrow{BD}\right)=\vec{0}\Rightarrow$ coplanar

4. i) x = 1 + 3t, y = -2+7t, z = 3 + 2t
   ii) $\left(\frac{13}{5},\frac{23}{5},0\right)$
   iii) –x – 2y – 2z = -4

5. i) Area = $\frac{1}{2}\sqrt{75}$ unit^2
   ii) -3x + 2y + z = $-\frac{11}{2}$

6. i) Shown
   ii) 119.5^0
   iii) -12x +21y – 9z = 24

7. i) 2x +5y –z = -3
   ii) $\left(\frac{23}{15},-\frac{2}{3},\frac{41}{15}\right)$

8. i) $\frac{1}{\sqrt{70}}\langle -3,6,-5\rangle$
   ii) $\frac{9}{7}\langle 1,2,3\rangle$
   iii) 2 unit^3
   iv) x = 3 + t, y = 2t – 1, z = 3t + 1.

9. i) (2, 3, 0)
   ii) -11x + 5y + 7z = -7

10. i) 119.5^0
    ii) -12x - 15y – 9z = -12

11. i) $x = 3 - t, \quad y = 8 + t, \quad z = 4 - 3t$
    ii) $(0, 11\ {-5})$

12. i) $5x - 3y + 4z = 6$
    ii) $\left(-\frac{22}{13}, \frac{20}{13}, \frac{62}{13}\right)$

13. i) $\overrightarrow{AB} = \langle 3, -1, -6 \rangle$, $\overrightarrow{AC} = \langle 6, 1, -8 \rangle$
    ii) Area = 10.26 unit^2
    iii) $14x - 12y + 9z = 13$

14. i) $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if $\vec{a} \bullet (\vec{b} \times \vec{c}) = 0$
    $\lambda = -\frac{1}{2}$
    ii) Volume = $-7 - 14\lambda$ unit^3

15. $-\frac{52}{5}x - \frac{26}{5}y - \frac{52}{5}z = -\frac{52}{5}$

16. a) $x = 3t + 3, \quad y = -2t - 1, \quad z = 4t$
    b) $(2, -4, -1)$
    c) $\vec{v}_1 \bullet \vec{v}_2 = 0 \Rightarrow L_1$ orthogonal to L_2

17. i) $\vec{M} = \langle -10, 26, 16 \rangle$
    ii) $\hat{M} = \frac{1}{\sqrt{1032}} \langle -10, 26, 16 \rangle$
    iii) $\text{proj}_{\vec{S}} \vec{R} = \frac{3}{11} \langle 2, 5, -2 \rangle$
    iv) 113.62^0

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1. Expand $\sin x$ as a series of ascending powers of $\left(x-\frac{\pi}{6}\right)$ as far as the third term. Use this series to find the approximate value of $\sin 32^\circ$.

2. Find the binomial expansion of $(1+x)^n$ as far as x^3. Hence, use your expansion to find approximate value of $\sqrt{1.02}$ correct to 5 decimal places.

3. Show that $\ln\left(\frac{x}{x-1}\right)=-\ln\left(1-\frac{1}{x}\right)$. Given that $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$ find the expression of $\ln\left(\frac{x}{x-1}\right)$ as far as x^4. Use your expansion to find $\ln(1.25)$ correct to 4 decimal places.

4. Use Maclaurin's theorem to expand $\sin^2 x$ up to x^5. Hence, find
$$\int_0^1 \sin^2 x\,dx$$

5. Using the Maclaurin's series, expand the function $f(x)=x\tan^{-1}x$ in power of x as far as the term in x^4.

6. Using Taylor's series or Maclaurin's series, expand $\ln(1+x)$ in ascending powers of x as far as the term in x^6.
By writing $1+x+x^2=\frac{1-x^3}{1-x}$ and using the above result, show that
$$\ln\left(1+x+x^2\right)=x=\frac{1}{2}x^2-\frac{2}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5-\frac{1}{3}x^6+\ldots$$

7. Obtain the first four terms in the expansion of $f(x)=\frac{1}{3-x}$ in ascending powers of x. Using the result obtained, estimate the definite integral
$$\int_0^1 \frac{1}{3-x^3}\,dx$$

8. Show that $\cos x=\frac{1}{\sqrt{2}}\left[1-\left(x-\frac{\pi}{4}\right)-\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2+\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3+\ldots\ldots\right]$. Hence, use the expansion to approximate $\cos 47^0$.

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9. Expand $f(x)=(1+x)^n$ as power of x up to the term containing x^3 using the Maclaurin's theorem.

   i) Show that $(1-x^2)^{-\frac{1}{2}}=1+\frac{x^2}{2}+\frac{3x^4}{8}+\frac{5x^6}{16}+\ldots$

   ii) If $\sin^{-1}x=\int_0^x \frac{1}{\sqrt{1-x^2}}dx$, hence, obtain the series of $\sin^{-1}x$.

10. Obtain the first four terms in the expansion of $\frac{1}{3-x}$ in ascending powers of x. Use the result obtained, estimate $\int_0^1 \frac{1}{3-x}dx$ correct to three decimal places.

11. Expand $\cos x$ as a series of ascending powers of $\left(x-\frac{\pi}{2}\right)$ as far as the third term.

12. Obtain the Maclaurin expansion of $(1-x)^{-\frac{1}{3}}$ as far as the term in x^3.

    Hence estimate $\int_0^{\frac{1}{2}} \frac{1}{\sqrt[3]{1-x}}dx$ to three decimal places.

13. Obtain the Taylor expansion of the function $f(x)=x\ln x$ as far as the term in $(x-1)^4$. Use this expansion to approximate f(1.1) correct to four decimal places.

14. (i) Write out the Maclaurin series for $\sin\frac{1}{2}x$ as far as the term in x^3.

    (ii) Differentiate the series to obtain the Maclaurin series for $\cos\frac{1}{2}x$.

    (iii) Given that $\sqrt{1+\sin x}=\sin\frac{1}{2}x+\cos\frac{1}{2}x$, obtain the Maclaurin expansion for $\sqrt{1+\sin x}$

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Answer :

1. $\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2$,

   $\sin 32^o = 0.5424$

2. $(1+x)^n = 1 + nx + (n-1)n\frac{x^2}{2!} + (n-2)(n-1)n\frac{x^3}{3!}$

   $\sqrt{1.02} = 1.00875$

3. $\ln\left(\frac{x}{x-1}\right) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \frac{1}{4x^4}$

   $\ln(1.25) = 0.2230$

4. $\sin^2 x = x^2 - \frac{x^4}{3}$

   $\int_0^1 \sin^2 x\,dx = 0.26667$

5. $x\tan^{-1} x = x^2 + \frac{1}{6}x^3$

6. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6}$

7. $\frac{1}{3-x} = \frac{1}{3} + \frac{1}{9}x + \frac{1}{27}x^2 + \frac{1}{81}x^3$

   $\int_0^1 \frac{1}{3-x^3}\,dx = 0.3676$

8. $\cos 47^o = 0.682$

9. $(1+x)^n = 1 + nx + (n-1)n\frac{x^2}{2!} + (n-2)(n-1)n\frac{x^3}{3!}$

   i. Shown

   ii. $\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{56}$

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10. $\dfrac{1}{3-x}=\dfrac{1}{3}+\dfrac{1}{9}x+\dfrac{1}{27}x^2+\dfrac{1}{81}x^3$

$\displaystyle\int_0^1 \frac{1}{3-x^3}\,dx = 0.404$

11. $\cos x = -\left(x-\dfrac{\pi}{2}\right)+\dfrac{1}{3!}\left(x-\dfrac{\pi}{2}\right)^2-\dfrac{1}{5!}\left(x-\dfrac{\pi}{2}\right)^5$

12. $(1-x)^{-1/3}=1+\dfrac{1}{3}x+2x^2+\dfrac{14}{3}x^3$

$\displaystyle\int_0^{1/2} \frac{1}{\sqrt[3]{1-x}}\,dx = 0.726$

13. $x\ln x=(x-1)+\dfrac{(x-1)^2}{2}-\dfrac{(x-1)^3}{6}+\dfrac{(x-1)^4}{12}$

$f(1.1)=0.1048$

14. i. $\sin\dfrac{1}{2}x=\dfrac{1}{2}x-\dfrac{1}{48}x^3$

ii. $\cot\dfrac{1}{2}x=\dfrac{1}{2}-\dfrac{1}{16}x^2$

iii. $\sqrt{1+\sin x}=\dfrac{1}{2}+\dfrac{1}{2}x-\dfrac{1}{16}x^2-\dfrac{1}{48}x^3$

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